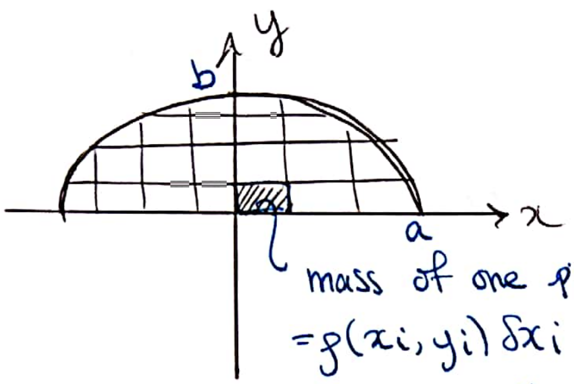
MA20223 PS1 2019-20

$\boxed{Q1}$ a) Explain Riemann sum interpretation.



mass of one piece $= \rho(x_i, y_i)\,\delta x_i\,\delta y_i$ for (x_i, y_i) some point in the $\delta x_i\,\delta y_i$ rectangle

Riemann sum for mass $= \iint_S \rho(x,y)\cdot dx\cdot dy$

$$= \sum_{i=1}^{N} \rho(x_i, y_i)\,\delta x_i\,\delta y_i \quad , N \to \infty$$

where $\delta x_i \delta y_i$ are a rectangular discretisation of the ellipse using $i = 1, \dots, N$ pieces.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \dfrac{x^2}{a^2}}$ for top half

$$\therefore M = \int_{x=-a}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} y \cdot dy \cdot dx$$

$$= \int_{-a}^{a} \frac{1}{2} b^2 \left(1 - \frac{x^2}{a^2}\right) \cdot dx = \frac{2ab^2}{3}$$

Similarly, we compute in other order.

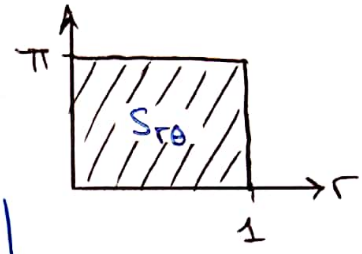
Ellipse: $x = \pm a\sqrt{1 - y^2/b^2}$ for $0 \le y \le b$.

$$M = \int_{y=0}^{b} \int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} y \cdot dx \cdot dy$$

$$M = \int_0^b 2ay\sqrt{1 - y^2/b^2} \cdot dy = \frac{2ab^2}{3}$$

(c) We alternatively integrate via

$$x = ar\cos\theta \qquad 0 \le r \le 1$$
$$y = br\sin\theta \qquad 0 \le \theta \le \pi$$



Then $J = \left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$

Thus $M = \displaystyle\int_{r=0}^{1} \int_{\theta=0}^{\pi} y \cdot J \cdot d\theta \cdot dr$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\pi} (br\sin\theta)\, abr \cdot d\theta \cdot dr$$

$$= ab^2 \left(\int_0^1 r^2\, dr\right)\left(\int_0^\pi \sin\theta\right)$$

$$M = \frac{2ab^2}{3}$$

$\boxed{Q2}$ $\int_C \underline{F} \cdot d\underline{r}$ where $\underline{F} = (x, -y, z)$

(i) $\underline{r}(t) = (\cos 2\pi t, \sin(2\pi t), 4t)$, $t \in [0,1]$.

$$I = \int_{t=0}^{t=1} \begin{pmatrix} \cos 2\pi t \\ -\sin 2\pi t \\ 4t \end{pmatrix} \cdot \begin{pmatrix} -2\pi \sin 2\pi t \\ 2\pi \cos 2\pi t \\ 4 \end{pmatrix} dt = \underline{\underline{8.}}$$

(ii) $\underline{r}(t) = (1,0,0) + t(0,0,4)$

$$I = \int_{t=0}^{1} \begin{pmatrix} 1 \\ 0 \\ 4t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} dt = \underline{\underline{8.}}$$

(b) Need ϕ s.t. $\underline{F} = \nabla\phi \Rightarrow$ Solve $\begin{cases} \frac{\partial \phi}{\partial x} = x \\ \frac{\partial \phi}{\partial y} = -y \quad \frac{\partial \phi}{\partial z} = z \end{cases}$

Gives $\underline{\underline{\phi = \frac{1}{2}(x - y^2 + z^2) + \text{const.}}}$

By FTC, $\int_C \underline{F} \cdot d\underline{r} = \phi(1,0,4) - \phi(1,0,0)$

$= \underline{\underline{8}}$ as expected.

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$\boxed{Q3}$ (a) Prove $\int_C f\,ds = \int_{-C} f\,ds$

We have $-C$ curve given by $\tilde{r}(t) = r(-t)$, $t \in [-b, -a]$

So $\int_{-C} f\,ds = \int_{t=-b}^{-a} f(\tilde{r}(t)) \left| \frac{d\tilde{r}}{dt} \right| \cdot dt$

But $\frac{d}{dt}\tilde{r}(t) = -r(-t)$ by the chain rule.

Hence,

$$\int_{-C} f\,ds = \int_{t=-b}^{t=-a} f(r(-t)) \left| \frac{dr(-t)}{dt} \right| dt$$

Let $\tilde{s} = -t \Rightarrow ds = -dt$.

Then $\int_{-C} f\,ds = \int_{\tilde{s}=a}^{\tilde{s}=b} f(r(\tilde{s})) \left| \frac{dr}{d\tilde{s}} \right| \cdot d\tilde{s}$.

$$= \int_C f\,ds \qquad \square$$

(b) Exactly same as (a) except that the magnitude $|\ldots|$ doesn't kill the sign $\Rightarrow$ answer will differ by negative.

Q4

(a) $\underline{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

verify easily that $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ is such that $\nabla\phi = \underline{F}$ by rules of differentiating $\tan^{-1}$.

(b). Let C be described by $\underline{r}(\theta) = (a\cos\theta, a\sin\theta, 0)$
$\theta \in [0, 2\pi]$.

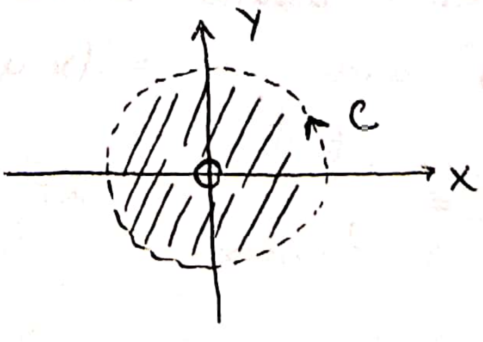
$$I = \int_{\theta=0}^{2\pi} \left(-\frac{a\sin\theta}{a^2}, \frac{a\cos\theta}{a^2}, 0 \right) \cdot (-a\sin\theta, a\cos\theta, 0) \, d\theta$$

$$= \int_{\theta=0}^{2\pi} (1)\, d\theta = 2\pi$$

note answer $= -2\pi$ if contour is clockwise.

(c) Contradiction not true because ϕ is not differentiable at the origin $(0,0)$.

Thm. only applies if region is simply connected (no holes).



ϕ only defined on punctured region

$\boxed{Q5}$ Thm states that following is equivalent

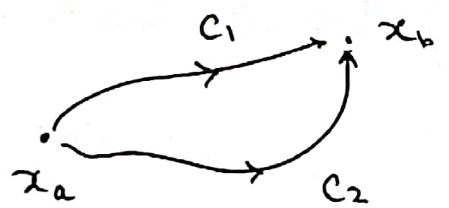
(1) $\underline{F}$ conservative on simply connected Ω

(2) $\oint_C \underline{F} \cdot d\underline{r} = 0$

(3) $\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{C_2} \underline{F} \cdot d\underline{r}$ if C_1, C_2 have same endpoints.

(a) (2) $\Rightarrow$ (3)

Pf: Consider closed contour C and pick two arbitrary points $\underline{x}_a$ and $\underline{x}_b$. ($\underline{x}_a \neq \underline{x}_b$).



- Then $C = C_1 \cup (-C_2)$
- By (2) $\int_C \underline{F} \cdot d\underline{r} = 0$.
- Thus $\int_{C_1} \underline{F} \cdot d\underline{r} + \int_{-C_2} \underline{F} \cdot d\underline{r} = 0$.

But by Q2, $\int_{-C_2} \underline{F} \cdot d\underline{r} = -\int_{C_2} \underline{F} \cdot d\underline{r}$

Hence $\int_{C_2} \underline{F} \cdot d\underline{r} = \int_{C_1} \underline{F} \cdot d\underline{r} \Rightarrow$ (3)

$\square$

(b) Show (3) $\Rightarrow$ (1).

Pf. As hint indicates, assume WLOG that

$\underline{x}_a = (0,0,0)$ and $\underline{x}_b = (x,y,z)$

Consider $\phi(\underline{x}) = \int_{(0,0,0)}^{(x,y,z)} \underline{F} \cdot d\underline{r}$

- We want to prove that $\nabla\phi = \underline{F}$ for an appropriate ϕ.
- By assumption $\phi(\underline{x})$ uniquely defined since it does not matter the path we take. When we define via

$$\phi(x,y,z) \equiv \phi_0 + \int_{(x_0,y_0,z_0)}^{(x,y,z)} \underline{F} \cdot d\underline{r}$$

We can take WLOG $\phi_0 \equiv 0$ and $(x_0, y_0, z_0) = (0,0,0)$

- Consider the contour C from $(0,0,0)$ to (x,y,z).
- Set $C = C_1 \cup C_2$

$$C_1 : \underline{r}_1(t) = (0,0,0) + t(0,y,z)$$
$$C_2 : \underline{r}_2(t) = (0,y,z) + t(x,0,0)$$

- Then

$$\phi(x,y,z) = \underbrace{\int_0^1 \underline{F}(\underline{r}_1(t)) \cdot \underline{r}_1'(t)\, dt}_{①} + \underbrace{\int_0^1 \underline{F}(r_2(t)) \cdot r_2'\, dt}_{②}$$

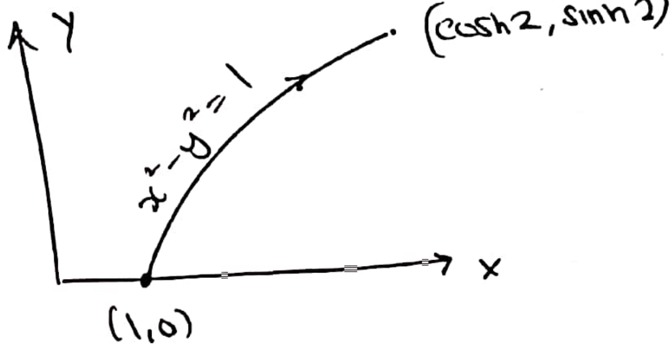
note that ① is independent of x, so $\frac{\partial ①}{\partial x} = 0$.

- We transform ② so integrand indep. of $x \Rightarrow$ let $s = tx$ so that $0 \leq s \leq x$ and $ds = x \cdot dt$

Then $② = \displaystyle\int_{s=0}^{s=x} \underline{F}(s,0,0) \cdot (1,0,0)\, ds = \int_{s=0}^{s=x} F_1(s)\, ds$

$\therefore \frac{\partial ②}{\partial x} = F_1(x)$ by FTC. $\therefore \frac{\partial \phi}{\partial x} = F_1(x)$. Other directions done similarly $\square$

Q6 Calculate $\int_C (3x^2 + 3y^2)^{1/2} \cdot ds$ where C is given by:



Let $\underline{r}(t) = (\cosh t, \sinh t) \Rightarrow \underline{r}'(t) = (\sinh t, \cosh t)$

note $\cos^2 t - \sinh^2 t = 1$

$ds = |\underline{r}'(t)|\, dt = \sqrt{\sinh^2 t + \cosh^2 t} \cdot dt$

Also $f(\underline{r}(t)) = 3^{1/2} (\sinh^2 t + \cosh^2 t)^{1/2}$

$$\therefore I = \int_{t=0}^{t=2} 3^{1/2} \left(\sinh^2 t + \cosh^2 t\right) dt$$

use $\sinh^2 t + \cosh^2 t = 2\cosh^2 t - 1 = \cosh 2t$

$$\therefore I = \sqrt{3} \cdot \int_0^2 \cosh 2t \cdot dt = \frac{\sqrt{3}}{2} \sinh 2t \Big|_0^2$$

$$= \frac{\sqrt{3}}{2} \sinh 4$$

$\boxed{Q7}$

(a) We do ABC and CBA done similarly.

Each segment : $C_1: \underline{r}_1(t) = (0,0) + t(2,0)$

$C_2: \underline{r}_2(t) = (2,0) + t(0,1)$

$C_3: \underline{r}_3(t) = (2,1) + t(-2,-1)$

with each going $0 \leq t \leq 1$.

$$\oint_C \underline{F} \cdot d\underline{r} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} \right) F(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

The three integrals are.

$$\int_{C_1} \underline{F}(\underline{r}_1(t)) \cdot \underline{r}_1'(t)\, dt = 4 \qquad \int_{C_2} \underline{F}(\underline{r}_2(t)) \cdot r_2'(t)\, dt = -\frac{13}{2}$$

$$\int_{C_3} \underline{F}(\underline{r}_3(t)) r_3'(t)\, dt = -\frac{13}{6}$$

The sum is thus $\oint_C \underline{F} \cdot d\underline{r} = -\frac{14}{3}$

(b) Seek ϕ s.t. $\nabla\phi = \underline{F} = (2x + y^2, 3y - 4x)$

$\phi_x = 2x + y^2 \Rightarrow \phi = x^2 + y^2 x + f(y)$

$\Rightarrow \phi_y = 2xy + f'(y)$

This must equal $3y - 4x$ which is not possible.

$\square$

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