LECTURE 1: (CHAP. 1): DERIVING THE TRANSPORT EQ.

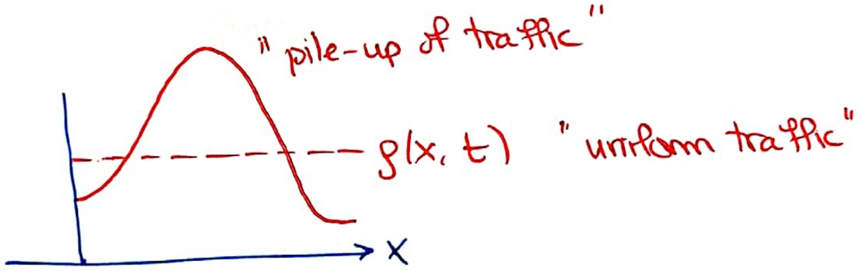
We want to motivate this course with an example.

Let $\rho(x,t)$ = density of cars at position x and time t

$[\rho]$ = # of cars per unit length.

units

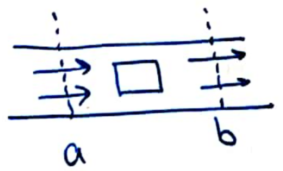
For example:



Let's consider a stretch of road $[a,b] \subset \mathbb{R}$.

(*) $\underbrace{\dfrac{\partial}{\partial t}\displaystyle\int_a^b \rho(x,t)\cdot dx}_{\text{rate of change of cars in } [a,b]}$ = cars in at $x=a$ − cars out at $x=b$.

$= q(a,t) - q(b,t)$



where $q(x,t)$ = flow (flux) of traffic.

and $[q]$ = # of cars per unit time

(*) $$\frac{\partial}{\partial t}\int_a^b \rho(x,t)\cdot dx = -\int_a^b \frac{\partial q}{\partial x}\cdot dx$$

$$\rightarrow \int_a^b \left(\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x}\right) dx = 0.$$

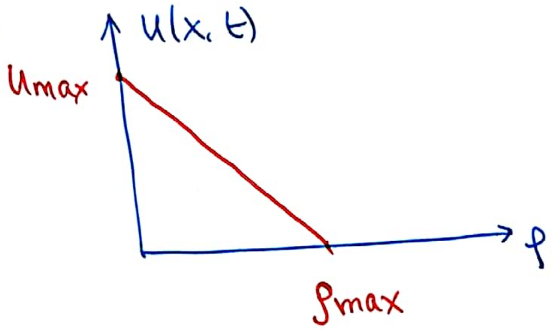
$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0}$$ since a, b is general

TRANSPORT EQUATION.

We need an empirical law for $q(x,t)$. Note

$$q(x,t) = \underbrace{\rho(x,t)}_{\text{\# of cars per m}} \times \big(\underbrace{\text{velocity}, u(x,t)}_{\text{m/s}}\big)$$

For example



e.g. $u = u_{max} - k\rho$ $\quad k$ = constant

$\therefore$ $\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0}$ where $q = \rho\{u_{max} - k\rho\}$.

to be solved for $\rho(x,t)$.

This seems very general + powerful. We could have equally modeled.

1) Traffic, populations, biological processes.
2) heat flow, water flow
3) Electricity and magnetism
4) Stocks.

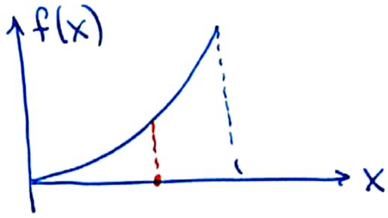
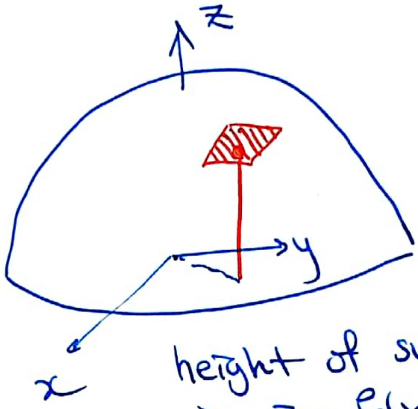
This inspires questions.

1) How do we extend to 3D (and beyond)
   Need vector calculus

2) How do we solve these equations
   (i) analytical methods
   (ii) numerical methods
   (iii) experimental methods $\quad$ THIS IS THE START.

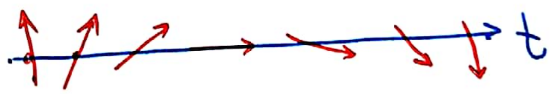
CHAPTER 2: REVIEW OF MULTIVARIABLE CALCULUS

The visualisation of functions is key.

FUNCTION.	VISUAL
1D Scalar, $f: \mathbb{R} \to \mathbb{R}$. e.g. $f(x) = x^2$	
2D Scalar, $f: \mathbb{R}^2 \to \mathbb{R}$. e.g. $f(x,y) = 1 - x^2 - y^2$	 height of surface is $z = f(x,y)$
3D Scalar, $f: \mathbb{R}^3 \to \mathbb{R}$. e.g. $f(x,y,z) = x^2 + y^2 + z^2$	need 4D space but can think of every (x,y,z) having a colour, temperature.

2D Vector on a line $\underline{v}: \mathbb{R} \to \mathbb{R}^2$

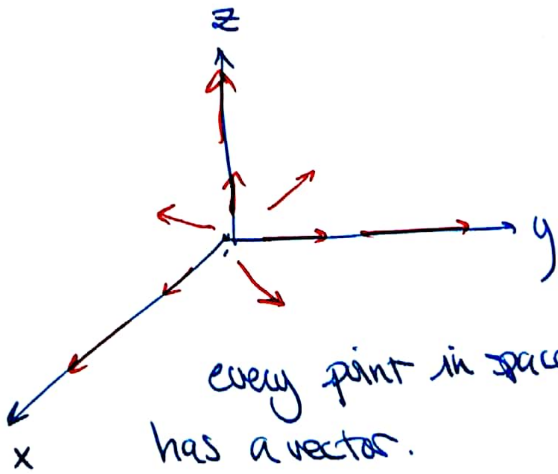
e.g. $\underline{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$



3D Vector in $\mathbb{R}^3$

e.g. $\underline{v}: \mathbb{R}^3 \to \mathbb{R}^3$.

$\cancel{\underline{v}(t)}$ $\underline{v}(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



every point in space has a vector.

§ 2.1 Double / Triple integrals

We continue our review of MVC.

① $I_1 = \int_a^b f(x) \cdot dx$

② $I_2 = \iint_\Omega f(x,y)\, dx dy = \int_\Omega f(x,y)\, dA$

$\Omega \subseteq \mathbb{R}^2$

③ $I_3 = \iiint_\Omega f(x,y,z)\, dx\, dy \cdot dz = \int_\Omega f(x,y,z)\, dV.$

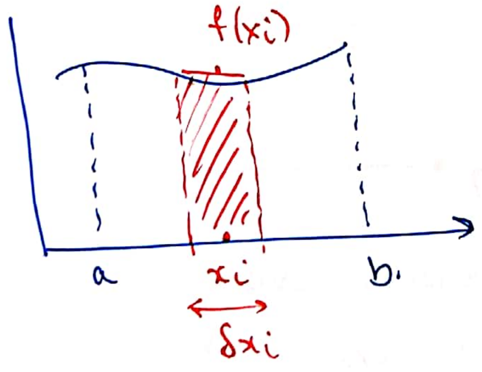
$\Omega \subseteq \mathbb{R}^3$

<u>NB</u>: We sometimes use $\iint$ vs. $\int$ and $\iiint$ vs. $\int$

Also shorthand $\left.\begin{array}{l} dA = dx\, dy \\ dV = dx\, dy\, dz. \end{array}\right\}$ Cartesian area elements.

It is important to visualise I_1, I_2, I_3 using a Riemann sum.

①



Define

$$I_1 \equiv \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i) \cdot \delta x_i$$

where $x_1, x_2, \dots, x_N$ subdivide $[a,b]$.

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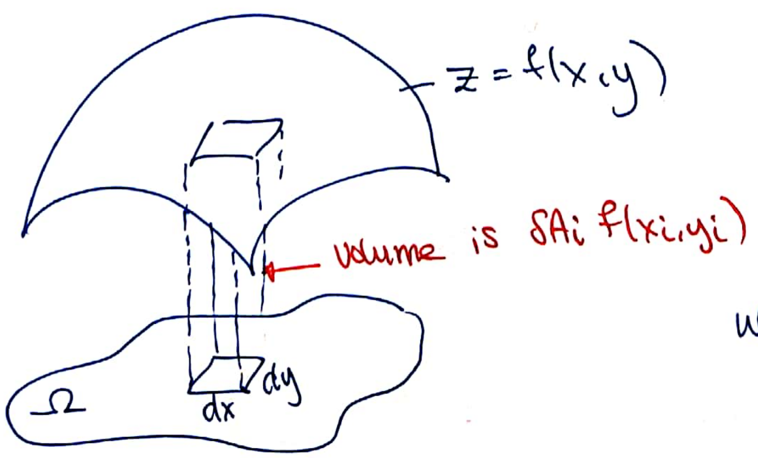
②



Similarly define

$$I_2 = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\, \delta A_i$$

where we subdivide Ω into $A_1, A_2, A_3, \dots, A_N$.

Thm 2.6: Given $x = x(u,v)$, $y = y(u,v)$ that maps Ω in (x,y) to Ω_{uv} in (u,v) then

$$\iint_{\Omega} f(x,y)\, dx \cdot dy = \iint_{\Omega_{uv}} f(x(u,v), y(u,v))\, J \cdot du \cdot dv \,.$$

where J = Jacobian

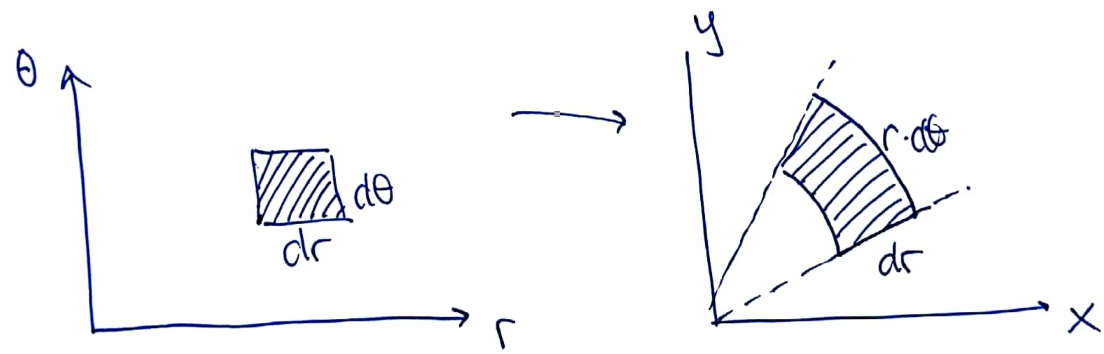
$$= \det \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

(partial derivs.)

Example 2.7

In polar coords, $x = r\cos\theta$, $y = r\sin\theta$,

then $J = r$.



The Jacobian indicates how area elements relate from one space to the other.

§ 2.2. Directional derivatives and gradient

<u>Def'n 2.13</u>: (Gradient) Let $\Omega \subseteq \mathbb{R}^3$, and $f : \Omega \to \mathbb{R}$. Then

$$\nabla f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = f_x \underline{i} + f_y \underline{j} + f_z \underline{k}$$

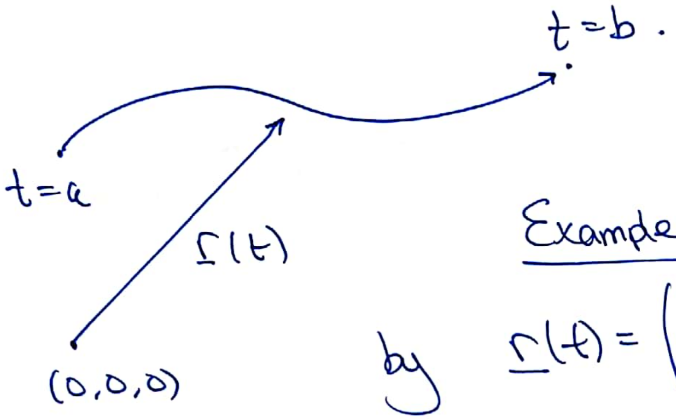
where $\underline{i}, \underline{j}, \underline{k}$ are unit vectors.

Gradient satisfies all usual linear properties (Lemma 2.14).

[We will come back to review § 2.2].

Chapter 3 : Line Integrals.

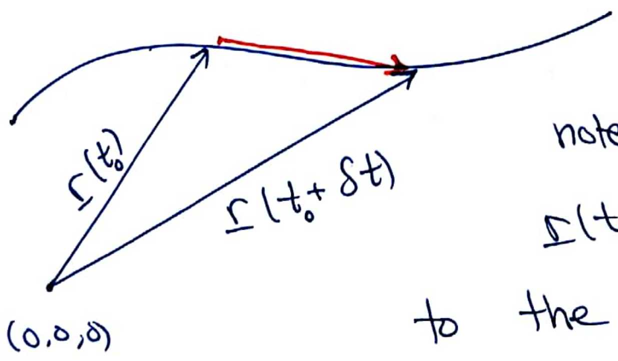
Def'n 3.1 (Curve) A curve in $\mathbb{R}^3$ is specified via a parameterisation $\underline{r} : [a,b] \to \mathbb{R}^3$



Example : A line in $\mathbb{R}^3$ is given by $\underline{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \underline{r}_0 + \underline{a} \cdot \underline{t}$

where $\underline{r_0}$ is initial point and $\underline{a}$ specifies the direction.

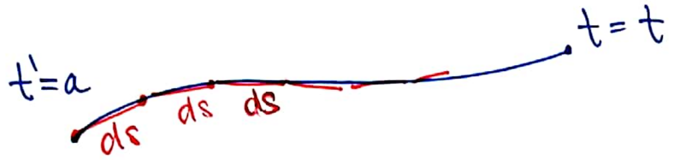
Lemma 3.3 (Tangent vector) $\frac{d\underline{r}}{dt}(t_0)$ is the tangent to the curve $\underline{r}(t)$ at $t = t_0$.



Pf by picture: note that $\underline{r}(t_0+\delta t) - \underline{r}(t_0)$ is parallel to the tangent as $\delta t \to 0$.

Lemma 3:4: (Arc length) The arclength of a curve from $t' = a$ to $t' = t$ is.

$$s(t) = \int_a^t \left| \frac{dr}{dt'} \right| dt'.$$



Note that we can write each infinitessimal element ds as,

$$ds^2 = dx^2 + dy^2 + dz^2$$

or $$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

Using $\underline{r}(t) = (x(t), y(t), z(t))$

$$\Rightarrow \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \cdot dt$$

$$= |\underline{r}'(t)| \cdot dt$$

and we have use the Riemann interpretation of $s(t) = \int_a^t ds$.

Example 3.5. Parameterisation of a circle.

We can write a circle as $\underline{r}(\theta) = (a\cos\theta, a\sin\theta)$

The circumference is then

$$I = \int_{\theta=0}^{\theta=2\pi} |\underline{r}'(\theta)| \cdot d\theta \qquad \underline{r}'(\theta) = (-a\sin\theta, a\cos\theta)$$

$$= \int_0^{2\pi} \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} \cdot d\theta = 2\pi a.$$

* note we had skipped

Def'n 3.2 (Simple & closed)

A curve is (i) simple if it does not intersect, i.e. $\underline{r}(t_1) \neq \underline{r}(t_2)$ if $t_1 \neq t_2$ and (ii) closed if $\underline{r}(a) = \underline{r}(b)$ for $t \in [a,b]$.

CHAPTER 3 CONTINUED.

Def'n 3.6 (Reversed curve $-C$)

Given curve $C \subset \mathbb{R}^3$ with paramet. $\underline{r}(t)$ for $t \in [a,b]$, then $-C$ is the curve given by. $\tilde{\underline{r}}(t)$ with $\tilde{\underline{r}}(t) = \underline{r}(-t)$

so $\hat{\underline{r}}(t)$ runs backwards.

§3.2 Line integrals of scalars.

Def'n 3.7 (Line integral of scalar)

Let C be given by $\underline{r}(t)$, $t \in [a,b]$. We define the line integral of $f(t)$ by

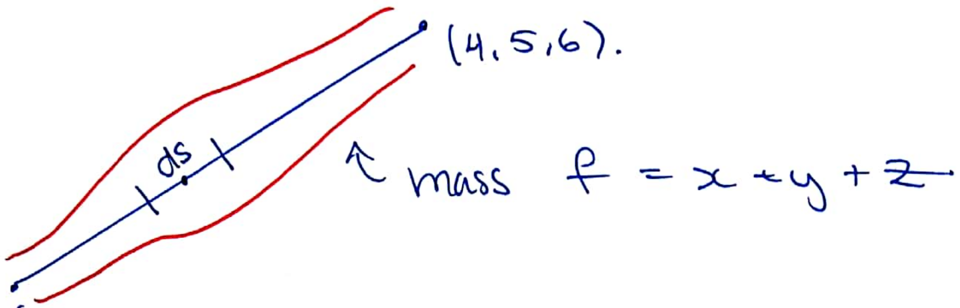
$$\int_C f \cdot ds \equiv \int_a^b f(\underline{r}(t)) |\underline{r}'(t)| \, dt$$

We also write $\oint_C f ds$ if C is closed.

Example 3.8. The mass per unit length along a curve given by the straight line from $(1,2,3)$ to $(4,5,6)$ is

$$f(x,y,z) = x + y + z.$$

What is the total mass?



The total mass is

$$M = \int_C f\,ds = \int_{t=a}^{t=b} f(\underline{r}(t))\,\|\underline{r}'(t)\|\,dt.$$

We parameterise by

$$\underline{r}(t) = (1,2,3) + (4-1,\,5-2,\,6-3)\,t$$
$$= (1,2,3) + (3,3,3)\,t$$

note that $t=0 \Rightarrow \underline{r}(0) = (1,2,3)$

$t=1 \Rightarrow \underline{r}(1) = (4,5,6)$

$$M = \int_{t=0}^{t=1} \left[(1+3t)+(2+3t)+(3+3t)\right] \times \left|(3,3,3)\right| \cdot dt = 3\sqrt{3}\left(6+\tfrac{9}{2}\right).$$

Lemma 3.9 gives basic properties of linearity, additivity, and reversability. [See problem set].

§3.3. Line of vector fields.

Def'n 3.10. (Vector of scalar integrals)

Let C be suff. nice curve and $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\int_C \underline{F}\, ds = \int_C \begin{pmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \\ F_3(\underline{x}) \end{pmatrix} ds = \begin{pmatrix} \int F_1\, ds \\ \int F_2\, ds \\ \int F_3\, ds \end{pmatrix}$$

$$\underline{x} = (x, y, z)$$

* not really used.

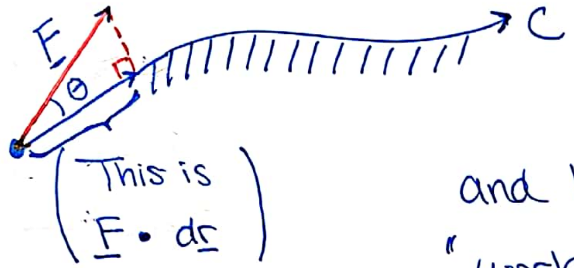
Really our focus is on what we call work integrals

Def'n 3.11 (Work integral of vector field)

Let C be suff. nice curve with $\underline{r}(t)$, $t \in [a,b]$ and let $\underline{F}$ be suff. nice vector field. The work integral is

$$\int_C \underline{F} \cdot d\underline{r} \equiv \int_{t=a}^{b} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt.$$

note that $\underline{F} \cdot d\underline{r}$ is the component of the field along the tangent to the curve since $\underline{F} \cdot d\underline{r} = |\underline{F}||d\underline{r}|\cos\theta$.



and hence $\underline{F} \cdot d\underline{r}$ is the "work" done at the given point.

<u>Example 3.12</u> Calculate the work integral for $\underline{F} = (3xy, -5z, 10x)$ and a curve $\underline{r}(t) = (t^2+1, 2t^2, t^3)$, $t \in [1,2]$.

$$\int_C \underline{F} \cdot d\underline{r} = \int_{t=1}^{t=2} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

$$= \int_1^2 \begin{pmatrix} 3(t^2+1)(2t^2) \\ -5t^3 \\ 10(t^2+1) \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} dt$$

$$= \cdots = 303.$$

Lemma 3.13 : This lemma covers the 3 basic properties of work integrals :

1) Linearity : $\int_C (\lambda \underline{F} + \mu \underline{G}) \cdot d\underline{r} = \lambda \int_C \underline{F} \cdot d\underline{r} + \mu \int_C \underline{G} \cdot d\underline{r}$

λ, μ scalar constants

2) Additivity : If $C = C_1 \cup C_2$ then

$$\int_C \underline{F} \cdot d\underline{r} = \int_{C_1} \underline{F} \cdot d\underline{r} + \int_{C_2} \underline{F} \cdot d\underline{r}$$

3) Non-independence of direction :

$$\int_{-C} \underline{F} \cdot dr = - \int_C \underline{F} \cdot d\underline{r}$$