CHAPTER 4 : The BIG theorem on conservative forces.

Def'n 2.20. (Conservative field and potential)

The field $\underline{F}$ is called conservative if $\exists$ some scalar potential, ϕ, defined on a simply connected region s.t. $\underline{F} = \nabla \phi$.

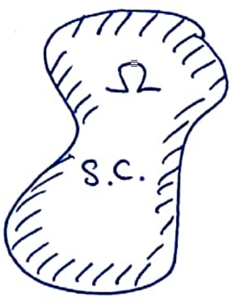
Def'n for simply connected happens in Chap. 4:

A set $\Omega \subset \mathbb{R}^3$ is simply connected if

(i) a path can be drawn between any two pts in Ω;

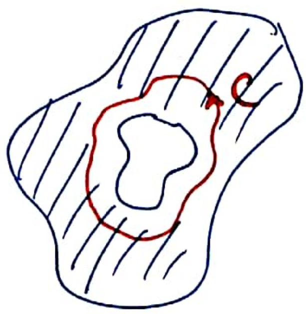
(ii) any closed curve $C \subset \Omega$ can be shrunk to a point.

(Def'n 4.2).



and not s.c...

and also not s.c.



As it turns out, many relevant physical models depend on conservative fields/forces.

Thm 4.1 : (Fundamental theorem of work integrals)
Let ϕ be a suff. smooth scalar field, and C be a curve with $\underline{r}(t)$, $t \in [a,b]$.
Then

$$\int_C \nabla\phi \cdot d\underline{r} = \phi(\underline{r}(b)) - \phi(\underline{r}(a)).$$

* note we think of $\underline{F} = \nabla\phi$ as the force.

Pf : Done by chain rule. Note.

$$\frac{d}{dt}\phi(\underline{r}(t)) = \frac{\partial\phi(\underline{r}(t))}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}(\underline{r}(t))\frac{dy}{dt} + \frac{\partial\phi(\underline{r}(t))}{\partial z}\frac{dz}{dt}$$

$$= \nabla\phi \cdot \underline{r}'(t).$$

But LHS of integral is.

$$\int_{t=a}^{t=b} \nabla\phi(\underline{r}(t)) \cdot \underline{r}'(t)\, dt = \int_a^b \frac{d}{dt}\phi(\underline{r}(t))\, dt =$$

$$= \phi(\underline{r}(b)) - \phi(\underline{r}(a))$$

(by FTC)

$\square$

* The lesson is that if we know a force $\underline{F}$ is conservative, and find ϕ s.t. $\underline{F} = \nabla\phi$, then $\int_C \underline{F} \cdot d\underline{r}$ is particularly simple.

The above shows also that $\int_C \underline{F} \cdot d\underline{r}$ is indep. of path if $\underline{F} = \nabla\phi$, i.e. it does not matter what C looks like. Is the the opposite direction true, i.e. to what extent can we guarantee

$$\exists\, \phi \text{ s.t. } \underline{F} = \nabla\phi \overset{?}{\iff} \int_C \underline{F} \cdot d\underline{r} \text{ is indep. of path.}$$

[See PS 1 homework Q].

The details are given by:

<u>Thm 4.3</u> (The BIG theorem on conservative forces)

The following statements are equivalent:

1. $\underline{F}$ is a conservative field on a simply connected domain Ω.
2. For every closed curve in Ω
$$\oint_C \underline{F} \cdot d\underline{r} = 0.$$
3. For any two curves C_1 and C_2 in Ω, both having the same start and end points
$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{C_2} \underline{F} \cdot d\underline{r}.$$
i.e. $\int \underline{F} \cdot d\underline{r}$ is indep of path.

<u>Pf</u>: (1) $\Rightarrow$ (2)

By (1) $\exists\ \phi$ s.t. $\underline{F} = \nabla\phi$ in Ω.

By FTC (work integrals),

$$\oint_C \underline{F} \cdot d\underline{r} = \phi(\underline{r}(b)) - \phi(\underline{r}(a)).$$

$= 0$ since $\underline{r}(b) = \underline{r}(a)$ for closed curves.

You'll do (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) in homework

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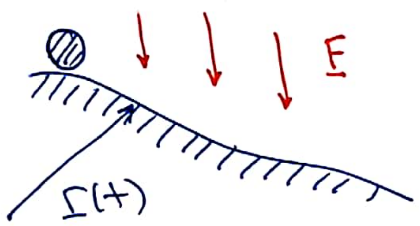
We want to link energy with $\int_C \underline{F} \cdot d\underline{r}$.

<u>Example 4.4</u> (Work = change in K.E.)



Let $\underline{r}(t)$ be the position vector of a particle of mass m under force $\underline{F}$. Show.

$$W = \int_C \underline{F} \cdot d\underline{r}$$

is equal to the change in kinetic energy of the particle, where kinetic energy $= \frac{1}{2} m v^2$, v = veloc.

(To do next lecture)

(Continued Example 4.4)

Note $\underline{r}'(t)$ = velocity of particle
$\underline{r}''(t)$ = acceleration of particle.

$$W = \int_C \underline{F} \cdot d\underline{r} = \int_C m\, r''(t) \cdot r'(t)\, dt \qquad (*)$$

note $$\frac{d}{dt}\left(\text{velocity}^2\right) = \frac{d}{dt}\left(\underline{r}'(t) \cdot \underline{r}'(t)\right)$$
$$= 2\underline{r}''(t) \cdot \underline{r}'(t)$$

by properties of the dot product (try yourself!)

So $(*) \Rightarrow$ $$W = \frac{1}{2}\int_{t=a}^{t=b.} m \frac{d}{dt}\left(|\underline{r}'(t)|^2\right) dt$$
$$= \frac{1}{2} m v^2(b) - \frac{1}{2} m v^2(a)$$
= change in kinetic energy. $\square$

Example 4.5 : Notice that if the force is conservative, then $\exists\, \phi$ s.t. $\underline{F} = \nabla\phi$, then

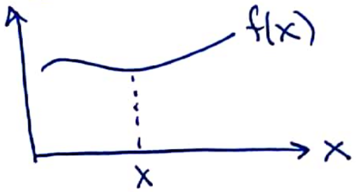
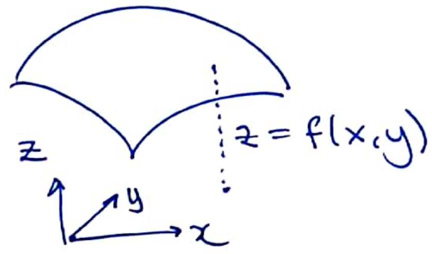
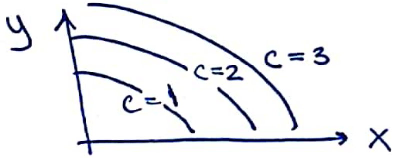
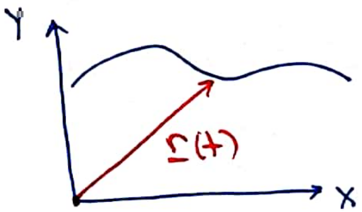
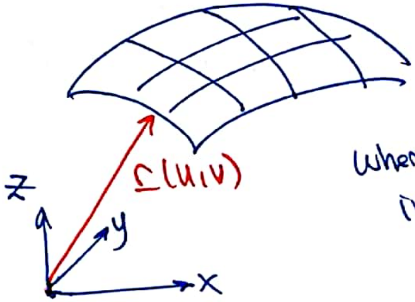
$$W = \int_C \underline{F} \cdot d\underline{r} = \int_C \nabla\phi \cdot dr. = \phi(\underline{r}(b)) - \phi(\underline{r}(a))$$
by F.T.C.

$\therefore$ Change in potential energy = Change in K.E.
$\phi(\underline{r}(b)) - \phi(\underline{r}(a))$ $\qquad$ $\frac{1}{2} m \{\underline{r}'(b)^2 - \underline{r}'(a)^2\}$.

(This is energy conservation.)

CHAPTER 5: Parameterisation of surfaces.

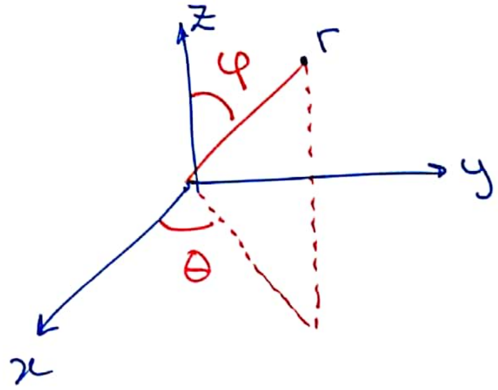
First we need to explain how to represent a surface.

	CURVE	SURFACE
EXPLICIT REPRESENTATION.	$y = f(x)$  e.g. $y = \sqrt{1-x^2}$	$z = f(x,y)$  $z = \sqrt{1-x^2-y^2}$
IMPLICIT. REPRESENTATION. (LEVEL SET)	$F(x,y) = c$  e.g. $x^2+y^2 = 1$.	$F(x,y,z) = c$  e.g. $x^2+y^2+z^2 = 1$
PARAMETRIC REPRESENTATION.	$\underline{r}(t) = (x(t), y(t))$ 	$\underline{r}(u,v) = (x(u,v), y(u,v), z(u,v))$  where (u,v) lie in some region of uv-plane.

Planar: $\underline{r}(\theta) = (\cos\theta, \sin\theta)$ $\quad 0 \le \theta \le 2\pi$

Surface: $\underline{r}(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ $\quad \theta \in [0, 2\pi)$, $\varphi \in [0, \pi)$

Note our spherical representation is for:



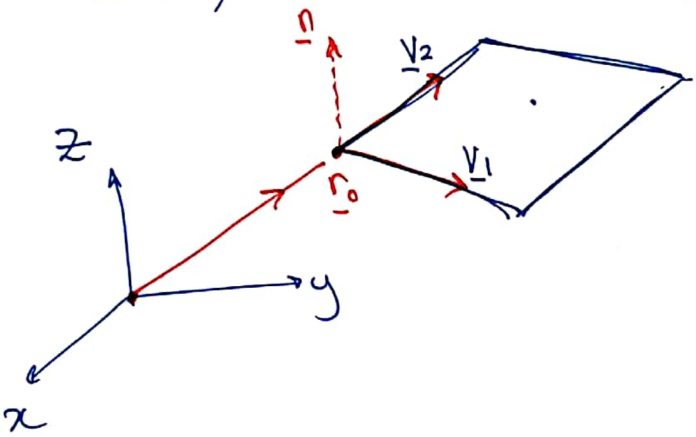
$0 \le \theta < 2\pi$
$0 \le \varphi < \pi$
$r = 1$ for unit sphere.

Our next step: Calculate normals of surfaces.

<u>Lemma 5.3</u> (Vector eqn of a plane)

(a) The eqn of a plane in $\mathbb{R}^3$ is

$$\underline{r}(\lambda,\mu) = \underline{r_0} + \lambda \underline{v_1} + \mu \underline{v_2}, \quad \lambda,\mu \in \mathbb{R}.$$



where $\underline{v_1}$ and $\underline{v_2}$ are not parallel.

(b) Moreover the unit normal to plane is,

$$\hat{\underline{n}} = \frac{\underline{v_1} \times \underline{v_2}}{|\underline{v_1} \times \underline{v_2}|} \; /$$

"hats = unit length"

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Lemma 5.4 (Normals to surfaces)

(a) Given surface S via $\underline{r}(u,v)$, the unit normal at $u=u_0$, $v=v_0$ is,

$$\hat{\underline{n}} = \frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|} \qquad \left(\begin{array}{l} \underline{r}_u = \dfrac{\partial \underline{r}}{\partial u} \\[2ex] \underline{r}_v = \dfrac{\partial \underline{r}}{\partial v} \end{array} \right)$$

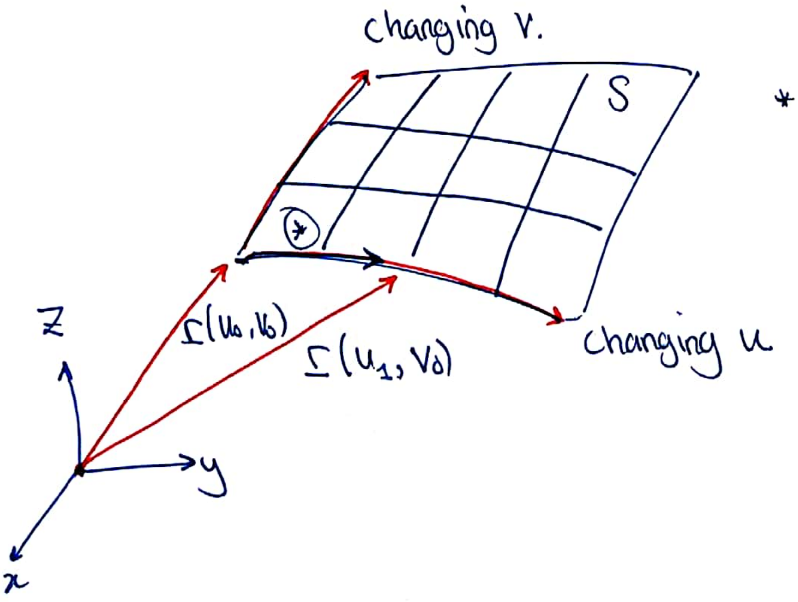
This gives $\hat{\underline{n}}$ for the parametric representation.

(b) If we have the implicit representation, $F(x,y,z) = c$, then

$$\hat{\underline{n}} = \frac{\nabla F}{|\nabla F|}$$

Pf:

(a) Note that $\underline{r}(u, v_0)$ for varying u and $\underline{r}(u_0, v)$ for varying v are curves passing through (u_0, v_0)



* note that this edge runs via $\underline{r}(u_1,v_0) - \underline{r}(u_0,v_0)$

(Pf of Lemma 5.4 continued) $\boxed{\text{LECTURE 6}}$

We see at point (u_0, v_0) the two vectors $\frac{\partial \underline{r}}{\partial u}$ and $\frac{\partial \underline{r}}{\partial v}$ run tangential to the surface.

Since e.g. $\lim_{u_1 \to u_0} \frac{\underline{r}(u_1, v_0) - \underline{r}(u_0, v_0)}{|u_1 - u_0|} = \frac{\partial \underline{r}}{\partial u}$.

$\therefore$ By Lemma 5.3 $\quad \hat{\underline{n}} = \frac{\left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right)}{\left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right|}$.

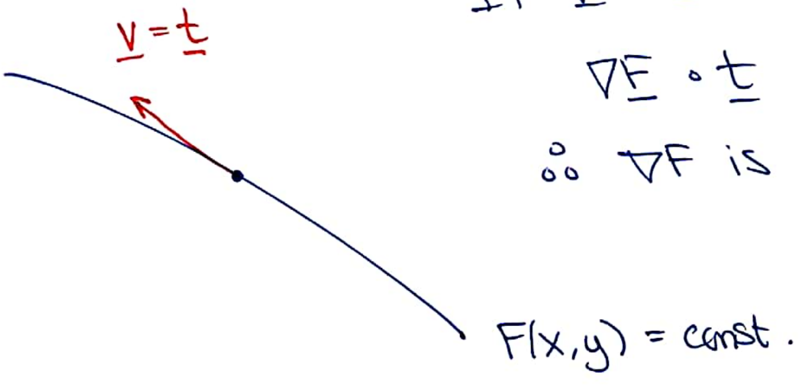
(b) This makes use of MVC Corollary 2.18: Given $F(x, y, z) = c$, then ∇F is perpendicular to the surface or level set.

This is because $\nabla F \cdot \underline{v}$ is the rate of change of F in the direction $\underline{v}$ (Directional derivative).

If $\underline{v} = \underline{t}$ = tangent, by def'n

$\nabla \underline{F} \cdot \underline{t} = 0$.

$\therefore \nabla F$ is $\perp$ to $\underline{t}$.



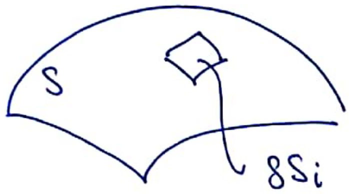
$\therefore \hat{\underline{n}} = \frac{\nabla F}{|\nabla F|}$ once we normalise.

(End of Chap. 5)

Chap. 6: Surface and Flux Integrals.

Want to define notion of surface integral:

$$\iint_S f(x,y,z)\, dS \equiv \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i,y_i,z_i)\, \delta S_i$$



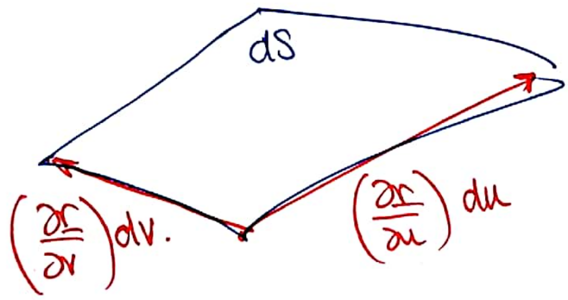
where S is a given surface and dS is a surface element.

Thm 6.2: Let S be a surface with paramet. $\underline{r}(u,v)$ where $(u,v) \in D$. Then

$$\iint_S f\, dS = \iint_D f(\underline{r}(u,v)) \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| . du\, dv.$$

Therefore $dS \equiv \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du . dv.$

Pf (Tuesday).



The result of $dS = |\underline{r}_u \times \underline{r}_v|\, du\, dv$ follows by area of parallelogram

In addition to (scalar) surface integrals, we have the analogy of the work integral.

Def'n 6.6. Let S be an orientable surface (where a normal can be defined) with outward pointing unit normal (OPUNV) $\hat{\underline{n}}$. and let $\underline{F}$ be a vector field.

The flux integral of $\underline{F}$ over S is.

$$\iint_S \underline{F} \cdot d\underline{S} \equiv \iint_S (\underline{F} \cdot \hat{\underline{n}})\, dS$$

* i.e. we have defined $d\underline{S} = \hat{\underline{n}}\, dS$.

Let's do examples of surface and flux integrals

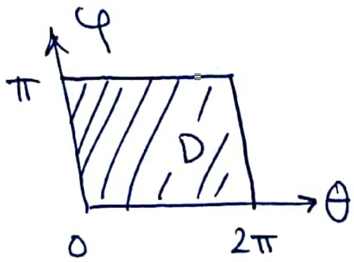
Example 6.4: Find the surface area of a sphere of radius a.

$$\text{Surface area} = \iint_S f(x,y,z)\, dS \quad \text{with } f \equiv 1.$$

Paramet. for S:

$\underline{r}(\theta,\varphi) = (a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ \cos\varphi)$

where $\theta \in [0, 2\pi)$, $\varphi \in [0, \pi)$



Then,

$$dS = \underline{|(r_\theta \times r_\varphi)|}\, d\theta d\varphi = \underline{|(-a\sin\theta\sin\varphi,\ a\cos\theta\sin\varphi,\ 0) \times (a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, \boxed{-a\sin\varphi})|}\, d\theta d\varphi$$

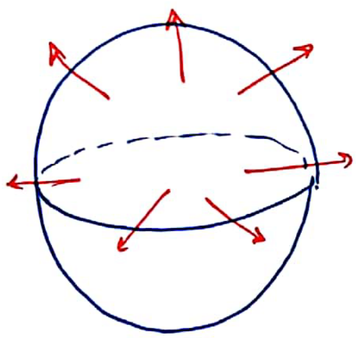
$$= \dots = a^2 \sin\varphi \; d\theta\, d\varphi$$

$$\text{Surface area} = \iint_D a^2 \sin\varphi \; d\theta\, d\varphi$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} a^2 \sin\varphi \; d\varphi \cdot d\theta.$$

$$= 4\pi a^2.$$

Example 6.7: Calculate the flux integral $\iint_S \underline{F} \cdot d\underline{S}$ where $\underline{F} = (x, y, z) = \underline{x}$ and S is the sphere of radius a and centre $(0,0,0)$.



We need $\iint_S \underline{F} \cdot \underline{\hat{n}}\, dS \equiv I$

We could do $I = \iint_D \underline{F} \cdot \Bigg(\underbrace{\frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|}}_{\underline{\hat{n}}} \Bigg) \underbrace{|\underline{r}_u \times \underline{r}_v|\, du\, dv}_{dS}.$

we could do this using above calculation.

Easier to note the normal to a sphere at the point (x, y, z) is just

$$\hat{\underline{n}} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = \frac{\underline{x}}{|\underline{x}|}$$

$$\therefore \; I = \iint_S \underline{x} \cdot \left(\frac{\underline{x}}{|\underline{x}|} \right) dS$$

$$= \iint_S |\underline{x}| \, dS = a \iint_S dS = a(4\pi a^2)$$