Justification of surface integral

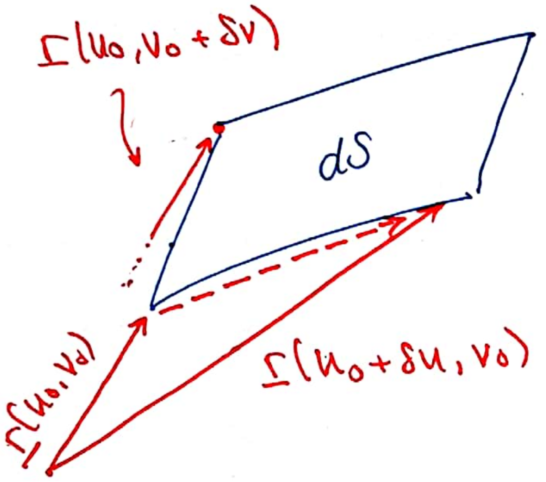
$\boxed{\text{LECTURE } 7}$

We found

$$\iint_S F \, dS = \iint_D F(\underline{r}(u,v)) \, dS$$

where $dS = \left| \underline{r}_u \times \underline{r}_v \right| du \, dv$.

and F is scalar.



note that

$$\underline{r}(u_0+\delta u, v_0) - \underline{r}(u_0, v_0)$$

$$\approx \left(\underline{r}(u_0,v_0) + \left.\frac{\partial \underline{r}}{\partial u}\right|_{(u_0,v_0)} \delta u + \dots \right) - \underline{r}(u_0,v_0)$$

$$\approx \frac{\partial \underline{r}}{\partial u}(u_0,v_0) \, \delta u + \text{higher order terms}.$$

Doing the same for $v_0 + \delta v$ and taking $\delta u, \delta v \to 0$

and thus $dS \doteq \left| \underline{r}_u \times \underline{r}_v \right| \cdot du \, dv$. (taking limit)

We now play video for intepreting $\displaystyle\iint_S \underline{F} \cdot \underbrace{\hat{\underline{n}} \, dS}_{d\underline{S}}$

where we see that flux integrals measure mass transport through a surface.

CHAP 7: The divegence and curl.

We define the divergence and curl by treating the derivatives as an operator

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

Define divergence as

$$\nabla \cdot \underline{F} = (\partial_x, \partial_y, \partial_z) \cdot (F_1, F_2, F_3) \quad \text{where } \underline{F} : \Omega \to \mathbb{R}^3 \text{ and } \Omega \subset \mathbb{R}^3$$

$$= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right)$$

<u>Example</u>: Find $\nabla \cdot \underline{F}$ for $\underline{F}(x, y, z) = \underline{x} = (x, y, z)$

$$\nabla \cdot \underline{F} = 1 + 1 + 1 = 3.$$

Similarly consider $\underline{F} = (-y, x, 0)$

$$\nabla \cdot \underline{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0.$$

Note (via Lemma 7.3) that the divergence is a linear operator and:

$$\nabla \cdot (\lambda \underline{F} + \mu \underline{G}) = \lambda (\nabla \cdot \underline{F}) + \mu (\nabla \cdot \underline{G})$$

$$\nabla \cdot (\phi \underline{F}) = \nabla \phi \cdot \underline{F} + \phi (\nabla \cdot \underline{F})$$

Here λ, μ scalars, $\underline{F}$ $\underline{G}$ vectors
ϕ scalar function

Similarly, we define the curl of a field $\underline{F}$ by

$$\boxed{\nabla \times \underline{F}} = (\partial_x, \partial_y, \partial_z) \times (F_1, F_2, F_3)$$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \underline{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y}\right) + \underline{k}\cdot\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Example:

$$\nabla \times \underline{x} = (\partial_x, \partial_y, \partial_z) \times (x, y, z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix}$$

$$= (0, 0, 0)$$

$$\nabla \times (-y, x, 0) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix}$$

$$= (0, 0, 2) = 2\underline{k}$$

Lemma 7.7:

$$\nabla \times (\lambda \underline{F} + \mu \underline{G}) = \lambda(\nabla \times \underline{F}) + \mu(\nabla \times \underline{G})$$

$$\nabla \times (\phi \underline{F}) = \nabla \phi \times \underline{F} + \phi(\nabla \times \underline{F})$$

Continued look at divergence + curl of CHAP. 7.

LECTURE 8

* Part of Lemma 7.7.

$$\phi = \phi(x, y, z)$$
$$\underline{F} = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix}$$

$$\underbrace{\nabla \times (\phi \underline{F})}_{\vee} = (\partial_x, \partial_y, \partial_z) \times (\phi \underline{F})$$
$$= \underbrace{\nabla \phi \times \underline{F}}_{\vee} + \underbrace{\phi (\nabla \times \underline{F})}_{\vee}$$

To prove : $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \phi F_1 & \phi F_2 & \phi F_3 \end{vmatrix} = \ldots = \text{RHS}$

Note we call $\underline{F}$ incompressible if : $\boxed{\nabla \cdot \underline{F} = 0.}$ (Def'n 7.4)

and we call $\underline{F}$ irrotational if $\boxed{\nabla \times \underline{F} = \underline{0}}$

§.7.3 Second derivatives

We can combine $\nabla \cdot$, $\nabla \times$, and ∇ (gradient) to produce different quantities, and the most crucial is :

Def'n 7.9 . For a scalar f, the Laplacian is
$$\boxed{\nabla^2 f = \nabla \cdot (\nabla f)}$$

Note the Laplacian is,

$$\nabla \cdot (\nabla f) = (\partial x, \partial y, \partial z) \cdot (f_x, f_y, f_z)$$
$$= f_{xx} + f_{yy} + f_{zz}.$$

Following notes Lemma 7.10 gives additional properties, such as

$$\nabla \times (\nabla f) = 0.$$

which is proved directly. Thus here.

$$(\partial x, \partial y, \partial z) \times (f_x, f_y, f_z)$$
$$= \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ f_x & f_y & f_z \end{vmatrix}$$
$$= i(\underbrace{f_{zy} - f_{yz}}_{0}) - j(\underbrace{f_{zx} - f_{xz}}_{0}) + k(\underbrace{f_{yx} - f_{xy}}_{0})$$
$$= 0.$$

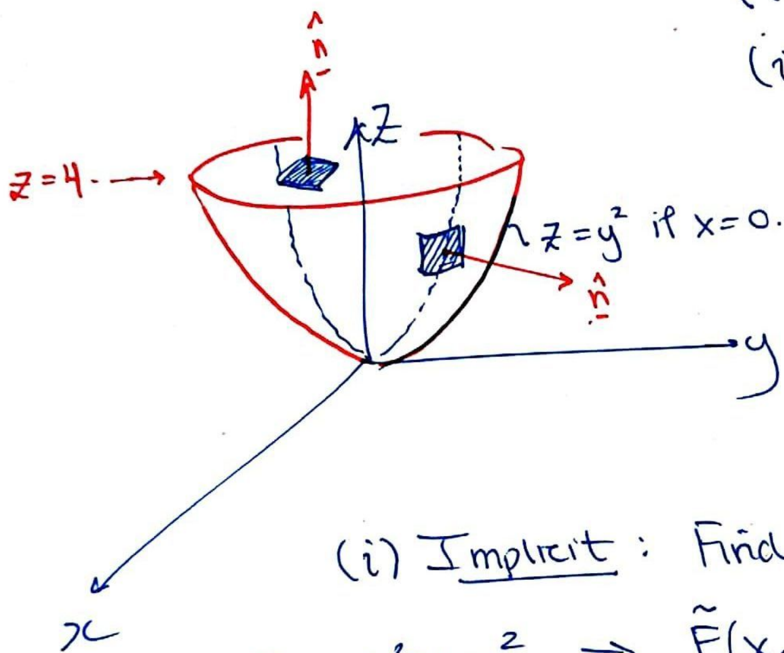
Example. Consider a paraboloid $z = x^2 + y^2$ (S) between $z = 0$ and $z = 4$, including the top.

(i) Calculate implicit and explicit representations of S and the normal

(ii) $I = \iint_S \underline{F} \circ \underline{n} \, dS$ where.

(iia) $\underline{F} = (x, y, 0)$

(iib) $\underline{F} = (x^2, y^2, 0)$



(i) Implicit: Find $\tilde{F}(x, y, z) = 0$.

$$z = x^2 + y^2 \Rightarrow \tilde{F}(x, y, z) = z - x^2 - y^2 = 0.$$

$$\therefore \quad \hat{n} = \frac{\nabla \tilde{F}}{|\nabla \tilde{F}|} = \frac{(-2x, -2y, 1)}{\sqrt{1 + (2x)^2 + (2y)^2}}$$

but take $\dfrac{(2x, 2y, -1)}{\sqrt{1 + 4x^2 + 4y^2}}$ for outer normal.

(ii) Write $\underline{r}(u, v)$ for explicit. representation.

$$\underline{r}(x, y) = (x, y, f(x, y))$$
$$= (x, y, x^2 + y^2)$$

Then $\hat{\underline{n}} = \dfrac{(\underline{r}_x \times \underline{r}_y)}{|\underline{r}_x \times \underline{r}_y|} = \dfrac{1}{|\ldots|} \begin{vmatrix} i & j & k. \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix}$

$$= \frac{(-2x, -2y, 1)}{\sqrt{1+4x^2+4y^2}}.$$

How about one last way? We could also set $x = r\cos\theta$, $y = r\sin\theta$, $z = x^2+y^2 = r^2$

and

$\underline{r}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ $\quad 0 \leq \theta < 2\pi$

$\quad 0 \leq r \leq 2.$

and $\hat{\underline{n}} = \dfrac{(\underline{r}_r \times \underline{r}_\theta)}{|\underline{r}_r \times \underline{r}_\theta)} = \dfrac{1}{|\ldots|} \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0. \end{vmatrix}$

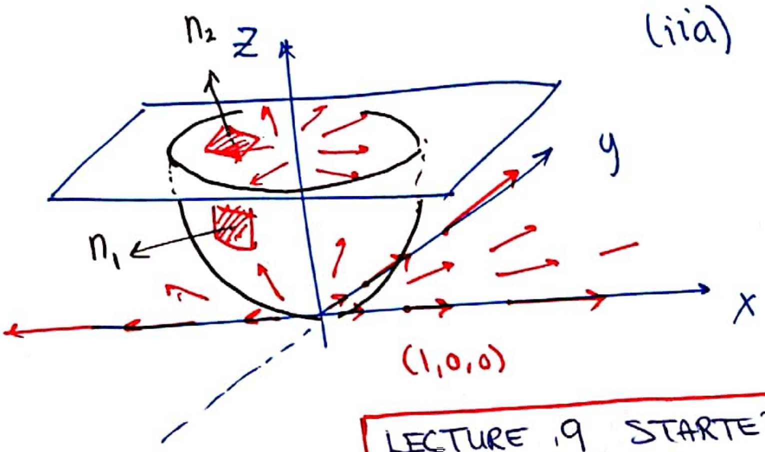
$$= \frac{1}{|\ldots|}\left(-2r^2\cos\theta, -2r^2\sin\theta, r^2\right)$$

$$= \frac{r\,(-2r\cos\theta, -2r\sin\theta, r)}{r\sqrt{r^2+2^2r^2}}$$

verify this is the same as above

(ii) We want to compute $I = \iint_S \underline{F} \circ \hat{\underline{n}} \, dS$.

Can we visualise?



(iia) $\underline{F} = (x, y, 0)$

From the pictures, we expect the integral on top surface $= 0$.

$\boxed{\text{LECTURE 9 STARTED HERE}}$

S_1 = side of paraboloid, $\hat{\underline{n}}_1$ = outward normal.

S_2 = top of paraboloid $z = 4$, $\underline{n}_2 = (0, 0, 1)$

$$I = \iint_{S_1} + \iint_{S_2} \underline{F} \circ \underline{n} \, dS = \left(\iint_{S_1} + \iint_{S_2} \right) \underline{F} \circ d\underline{S}$$

On S_2, $\underline{F} \circ \underline{n} = (x, y, 0) \circ (0, 0, 1) = 0$.

On S_1, use $\underline{r}(r, \theta) = (r\cos\theta, r\sin\theta, r^2)$

$$0 \leq \theta < 2\pi$$
$$0 \leq r \leq 2$$

$$I = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \underline{F}(\underline{r}(r, \theta)) \cdot \frac{(\underline{r}_r \times \underline{r}_\theta)}{|\underline{r}_r \times \underline{r}_\theta|} \underbrace{\left\{ |\underline{r}_r \times \underline{r}_\theta| \cdot dr \, d\theta \right\}}_{dS}$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} (r\cos\theta, r\sin\theta, 0) \cdot \underbrace{(\underline{r}_r \times \underline{r}_\theta) \, dr \cdot d\theta}_{d\underline{S}}.$$

Previously, we found $(\underline{r}_r \times \underline{r}_\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r^2)$

but must make sure to point outwards.

$$I = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (r\cos\theta, r\sin\theta, 0) \cdot (2r^2\cos\theta, 2r^2\sin\theta, -r^2) \, d\theta \cdot dr.$$

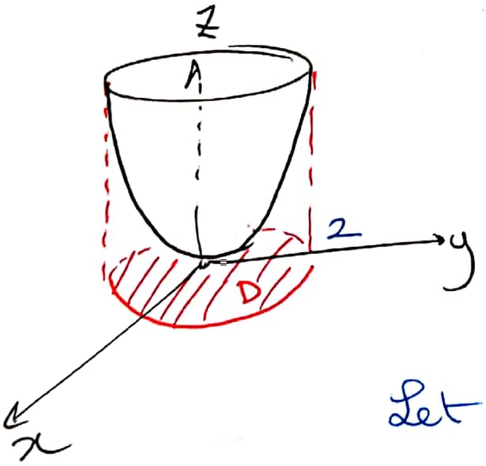
$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (2r^3\cos^2\theta + 2r^3\sin^2\theta) \cdot d\theta \cdot dr.$$

$$= \int_{r=0}^{2}\int_{0}^{2\pi} 2r^3 \, d\theta \cdot dr. = 2\pi \frac{2}{4} 2^4 = 2^4 \cdot \pi$$

ALTERNATIVE PARAMET.

Suppose we try $r(x,y) = (x, y, x^2+y^2)$.

Now $I = \iint_D \underline{F}(\underline{r}(x,y)) \cdot (\underline{r}_x \times \underline{r}_y) \, dx \, dy$.



Here D is the disc of radius 2 on the (x,y) plane.

$$\therefore I = \iint_D \underline{F}(\underline{r}(x,y)) \cdot (2x, 2y, -1) \, dx \, dy$$

Let $x = r\cos\theta$ ... reconnect.
$y = r\sin\theta$

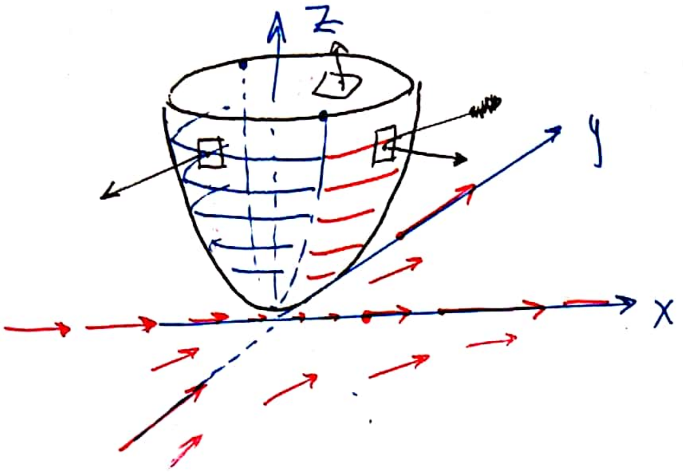
or $-2 \le x \le 2$
$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

but this is not so nice.

$$I = \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (r\cos\theta, r\sin\theta, 0) \cdot (2r\cos\theta, 2r\sin\theta, -1) \, \underbrace{dx \cdot dy}_{J \cdot dr \cdot d\theta \, = r \, dr \, d\theta}.$$

Scanned by CamScanner

(ii) b $\underline{F} = (x^2, y^2, 0)$



We expect that $\iint \underline{F} \cdot \underline{n} \, dS$ is zero on top and zero on sides.

$S_1 = \text{top} \Rightarrow \underline{n}_1 = (0,0,1), \quad \underline{F} \cdot \underline{n}_1 = 0.$

$S_2 = \text{side} \Rightarrow \underline{n}_2 = \dfrac{(\underline{r}_r \times \underline{r}_\theta)}{|\underline{r}_r \times \underline{r}_\theta|}$

$$I = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} (r^2\cos^2\theta, r^2\sin^2\theta, 0) \cdot (2r^2\cos\theta, 2r^2\sin\theta, r^2) \, dr\, d\theta$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} (2r^3\cos^3\theta + 2r^3\sin^3\theta) \cdot dr \cdot d\theta$$

use the fact $\int_0^{2\pi} \cos^3\theta \cdot d\theta = 0 = \int_0^{2\pi} \sin^3\theta \cdot d\theta$.

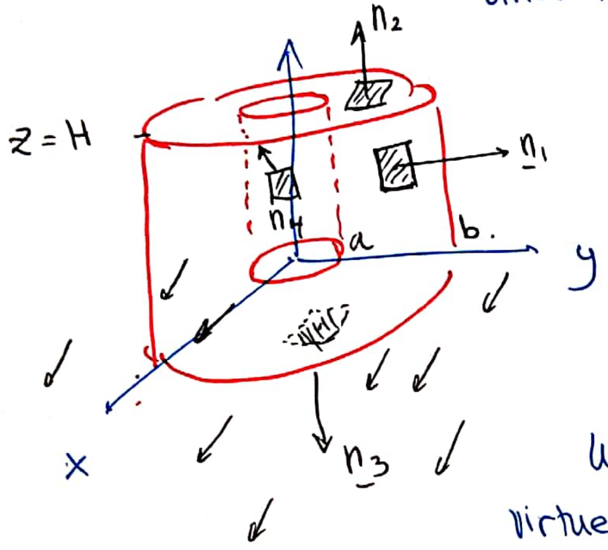
$\therefore I = 0.$

<u>Bonus</u>: Show S can be split into $S_1 \cup S_2$ where

$$\iint_{S_1} \underline{F} \cdot \underline{n} \, dS = -\iint_{S_2} \underline{F} \cdot \underline{n} \, dS.$$

Example : $\iint_S \underline{F} \cdot \underline{n} \, dS$ where $\underline{F} = (1, 0, 0)$

S = annular cylindrical region. shown below and including the base and top.



Calculate

$$I = \iint_{S_1 \cup \ldots \cup S_4} \underline{F} \cdot \underline{n} \, dS.$$

We find $I = 0$. by virtue of the symmetry of the problem.