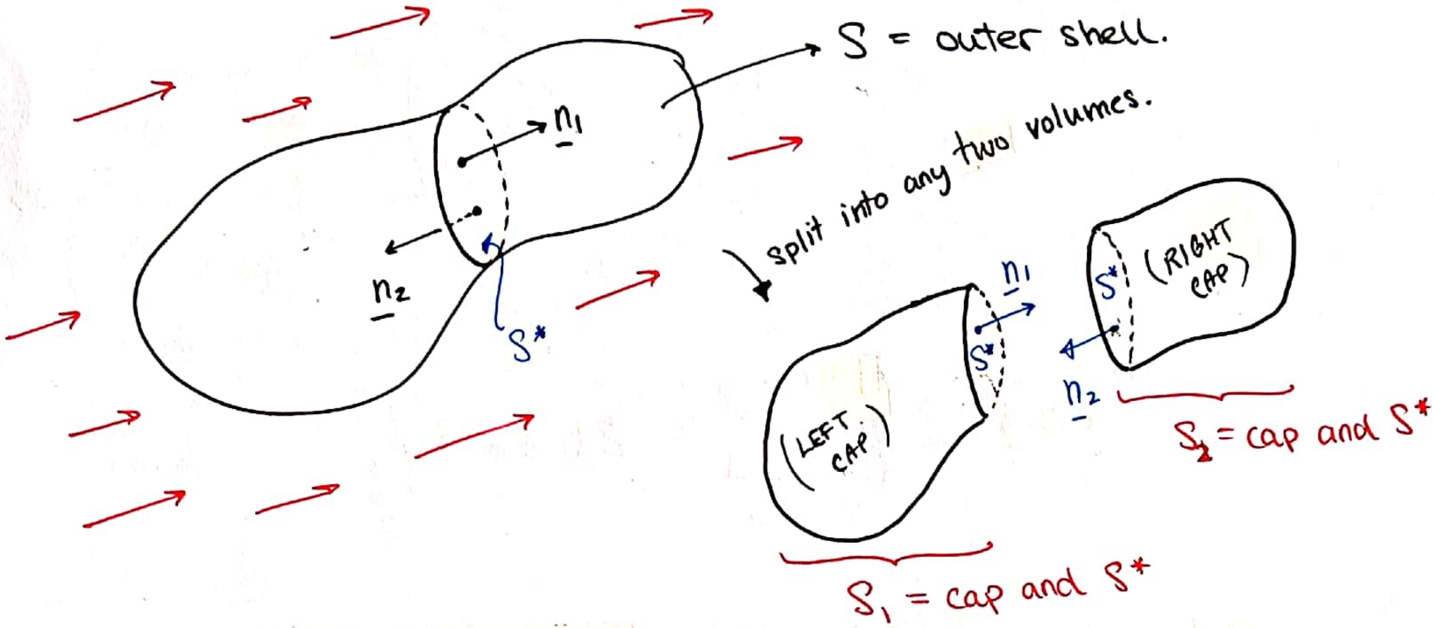
Examine the picture below:

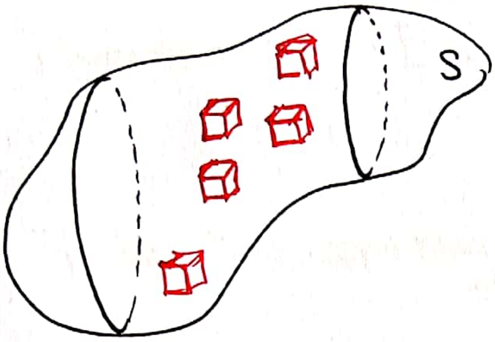


Note that $\displaystyle\iint_S \underline{F}\cdot\hat{\underline{n}}\, dS = \left(\iint_{S_1} + \iint_{S_2}\right) \underline{F}\cdot\hat{\underline{n}}\, dS$

because $\displaystyle\iint_{S^*\,(\text{left})} + \iint_{S^*\,(\text{right})} \underline{F}\cdot\hat{\underline{n}}\, dS = 0.$

and the left cap and right cap form S.

Let's split the surface into cuboids of volume $\Delta x \, \Delta y \, \Delta z$.

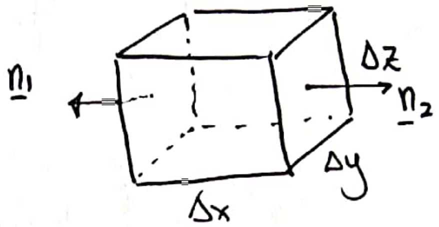


Then $$\iint_S \underline{F} \cdot \hat{\underline{n}} \, dS = \sum_i \iint_{S_i} \underline{F} \cdot \hat{\underline{n}} \, dS$$

Total flux = Sum of fluxes in little cuboids

What is the flux through a small box?

Consider $\iint_{box} \underline{F} \cdot \hat{\underline{n}} \, dS = \left(\iint_{S_1} + \dots + \iint_{S_6} \right) \underline{F} \cdot \hat{\underline{n}} \, dS$



Take two opposing sides:

$$\iint_{S_1} \underline{F} \cdot (-1,0,0) \, dS + \iint_{S_2} \underline{F} \cdot (+1,0,0) \, dS$$

$$= -F_1(x,y,z) \iint_{S_1} dS + F_1(x+\Delta x, y, z) \iint_{S_2} dS$$

$$\approx \frac{\partial F_1}{\partial x} \Delta x \, \Delta y \, \Delta z \quad \text{by Taylor's Theorem.}$$

Doing the same over all sides gives

$$\iint_S \underline{F} \cdot \hat{\underline{n}} \, dS = (\nabla \cdot \underline{F}) \, \Delta x \, \Delta y \, \Delta z .$$

divergence is related to flux of a tiny box!

Therefore

$$\iint_S \underline{F} \cdot \hat{\underline{n}} \, dS = \iiint_{\Omega} (\nabla \cdot \underline{F}) \, dV$$

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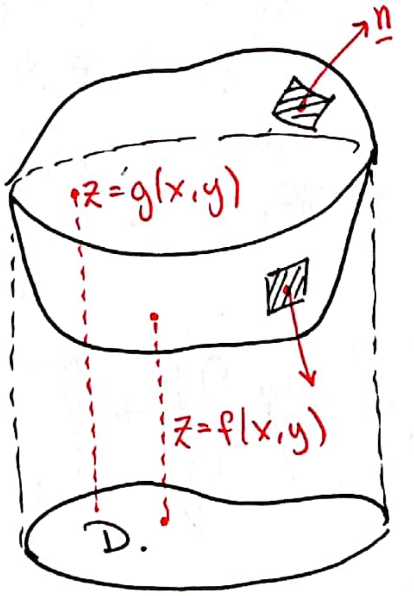
(THEOREM 8.6) DIVERGENCE THEOREM.

Let $\Omega \subset \mathbb{R}^3$ be bounded domain with a closed boundary $S = \partial\Omega$. Let $\underline{n}$ be outward unit normal. Then

$$\boxed{\iiint_{\Omega} \nabla \circ \underline{F} \, dV = \iint_{S} \underline{F} \circ \underline{n} \, dS.} \qquad (*)$$

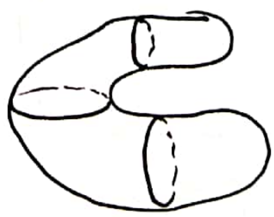
Pf.



Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D, \, f(x,y) \leq z \leq g(x,y)\}$.

This proof will work only for convex domains

This is not convex



Write (*)

$$\iiint_\Omega \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dV = \iint_S (F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}) \cdot \underline{n} \; dS .$$

(+) We want to show $\iiint_\Omega \frac{\partial F_3}{\partial z} dV = \iint_S F_3 \underline{k} \cdot \underline{n} \, dS$

(+) $\text{LHS} = \iint_D \int_{z=f(x,y)}^{g(x,y)} \partial F_3/\partial z \; dV$

$$= \iint_D [F_3(x,y,g(x,y)) - F_3(x,y,f(x,y))] \cdot dx \cdot dy .$$

(+) $\text{RHS} = \iint_{S^+} + \iint_{S^-} F_3 \underline{k} \cdot \underline{n} \, dS .$

On S^+ (top surface) $\underline{n} \, dS = (-g_x, -g_y, 1) \, dx \cdot dy$.

On S^- (bottom surface) $\underline{n} \, dS = (f_x, f_y, -1) \, dx \, dy$.

$$\therefore \text{RHS} = \underbrace{\iint_D F_3 \cdot dx \cdot dy}_{\text{on } z = g(x,y)} - \underbrace{\iint_D F_3 \cdot dx \cdot dy}_{\text{on } z = f(x,y)} = \text{LHS}$$

Similar task for x, y components. $\square$

Example 8.7. Verify that the divergence theorem holds for $\underline{F} = \underline{x} = (x, y, z)$.

and Ω = ball of radius a.

Show $\iiint_{\Omega} \nabla \cdot \underline{F} \, dV = \iint_{\partial\Omega} \underline{F} \cdot \underline{n} \, dS$

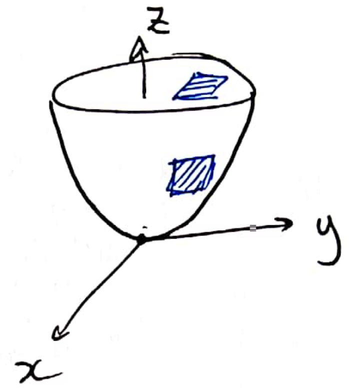
We already checked (Ex. 6.7) $\iint_{\partial\Omega} \underline{F} \cdot \underline{n} \, dS = 4\pi a^3$.

$$LHS = \iiint_{\Omega} 3 \cdot dV = 3\int_{r=0}^{a} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} J \, dr \cdot d\theta \, d\varphi$$

where $J = r^2 \sin\varphi$ $\quad = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$.

Example: Compute $\iint_S \underline{F} \circ \underline{n}\, dS$. for example in Wk 3 for paraboloid $z = x^2 + y^2$ and top $z = 4$.

(i) $\underline{F} = (x, y, 0)$
(ii) $\underline{F} = (x^2, y^2, 0)$



(i) $$I = \iiint_\Omega \nabla \circ \underline{F}\, dV$$

$$= \iiint_\Omega 2 \cdot dV$$

use cylindrical coords: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$; $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $x^2 + y^2 \le z \le 4$

$$I = 2 \cdot \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{z=r^2}^{4} J.\, dz \cdot d\theta \cdot dr.$$

where $J = r$. (Jacobian for cylindrical same as polar)

$$\therefore I = 2 \cdot (2\pi) \cdot \int_0^2 r \cdot (4 - r^2) \cdot dr.$$

$$= 2^2 \pi \left\{ \underbrace{2 \cdot 2^2 - \frac{1}{2^2} \cdot 2^4}_{4} \right\} = 2^4 \cdot \pi.$$

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(ii) $I = \iiint_{\Omega} (2x+2y) \cdot dV$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \int_{z=r}^{4} (2r\cos\theta + 2r\sin\theta)\, r \cdot dz \cdot d\theta \cdot dr.$$

but $\int_0^{2\pi} \cos\theta \cdot d\theta = 0 = \int_0^{2\pi} \sin\theta \cdot d\theta$

$\therefore I = 0$; which matches previous.

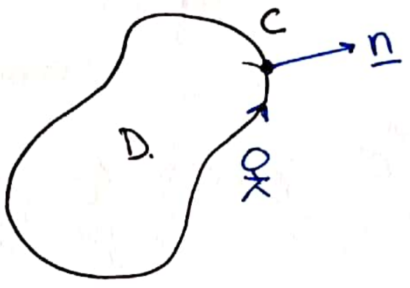
LECTURE 11.

CHAP. 8. GREEN'S THEOREM.

DEF'N 8.8. (POSITIVE ORIENTATION).

Let $D \subset \mathbb{R}^2$ be a bounded domain with closed boundary C.

Then C is oriented in the positive sense if, for given paramet. $\Gamma(t)$, $t \in [a,b]$, then we traverse C in an anti-clockwise manner as t increases



THM 8.9 (GREEN'S THM IN PLANE – DIVERGENCE VERSION)

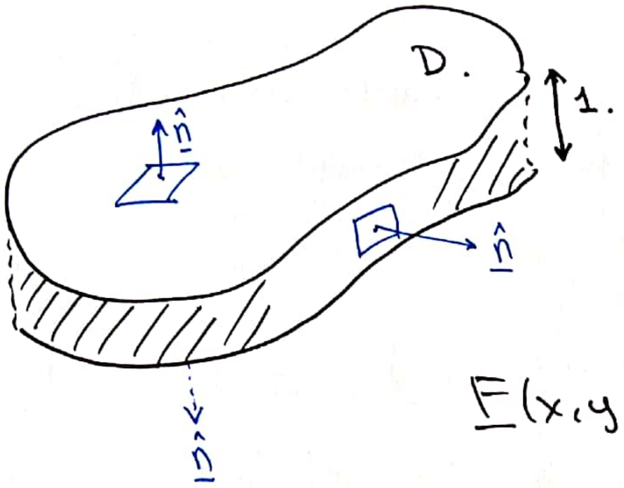
Let D be bounded domain in $\mathbb{R}^2$, C its boundary oriented in positive direction, with C non-intersecting (simple)

Then

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dA = \oint_C (p,q) \cdot \underline{n} \, ds.$$

for $p(x,y)$, $q(x,y)$ differentiable. Here $\underline{n}$ is the outward (unit) normal along C.

(Idea for proof)



Consider the volume:

$$V = \{(x,y,z) \mid (x,y) \in D, \; 0 \le z \le 1\}.$$

and consider

$$\underline{F}(x,y,z) = (p(x,y), q(x,y), 0).$$

Apply Div. Thm:

$$\iiint_V \nabla \cdot \underline{F} \, dV = \iint_S \underline{F} \cdot \underline{n} \, dS.$$

$$\text{LHS} = \iint_{(x,y)\in D.} \int_{z=0}^{z=1} \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) \cdot dz \cdot \underbrace{dA}_{dx\,dy}.$$

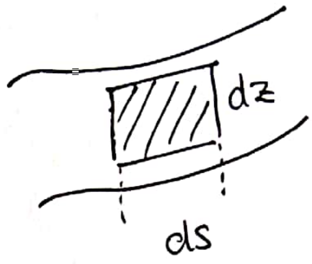
$$= \iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) \cdot dA.$$

$$\text{RHS} = \left(\iint_{\text{top}} + \iint_{\text{bottom}} + \iint_{\text{sides}} \right) \underline{F} \cdot \underline{n} \, dS$$

but $\underline{F} \cdot \underline{n} = 0$ on top and bottom.

$\therefore$ RHS $= \iint_{\text{Sides}} \underline{F} \cdot \underline{n} \, dS.$

* note that $dS = dz \cdot ds$ by geometry, or can be done by parameterisation.



Note that $\underline{n} = (n_1(x,y), n_2(x,y), 0)$

where (n_1, n_2) is the 2D normal.

$$\therefore \text{RHS} = \int_{z=0}^{z=1} \oint_C (p, q, 0) \cdot (n_1, n_2, 0) \, ds \cdot dz.$$

$$= \oint_C (p, q) \cdot \underline{n} \, ds$$

$\square$

THM 8.11 (GREEN'S THEOREM IN PLANE — STOKES' VERSION)

Let $D \subset \mathbb{R}^2$ be bounded domain with closed boundary C (simple and smooth), oriented in the positive sense. Suppose $\underline{F} = (F_1, F_2)$.

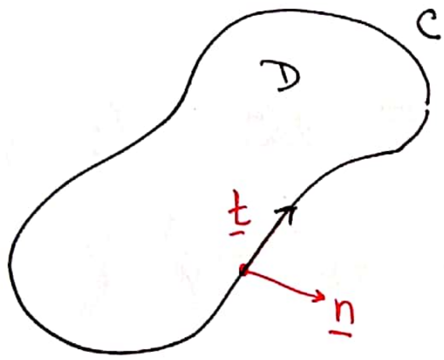
Then

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \underline{F} \cdot d\underline{r}$$

Pf. Use divergence version and let $(p,q) = (F_2, -F_1)$. Then we have

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C (F_2, -F_1) \cdot \underline{n}\, ds.$$

$C : \underline{r}(t) = (x(t), y(t))$, t = arclength. $= s$.



Recall tangential vector is

$$\underline{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$$

$$\underline{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$$

$$\text{RHS} = \oint_C (F_2, -F_1) \cdot \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) ds$$

$$= \oint_C (F_1, F_2) \cdot \underbrace{\left(\frac{dx}{ds}, \frac{dy}{ds}\right) ds}_{\underline{t}\, ds \,=\, d\underline{r}.}$$

$\square$

COROLLARY 8.13 (USE LINE INTEGRAL TO COMPUTE AREA).

Let C be simple, smooth curve in $\mathbb{R}^2$, oriented in the positive sense. The area enclosed by C is

$$\text{area of } D = \frac{1}{2} \cdot \oint_C (-y, x) \cdot d\underline{r}.$$

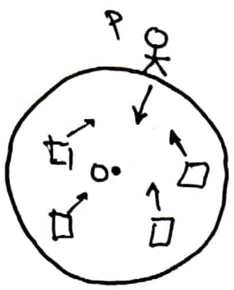
Pf. Let $\underline{F} = (-y, x)$ and apply Green's:

$$\oint_C \underline{F} \circ d\underline{r} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cdot dA$$

$$= \iint_D (1+1) \cdot dA = 2 \iint_D dA = 2 \cdot \text{Area}.$$

$\square$.

This Q. troubled Newton:



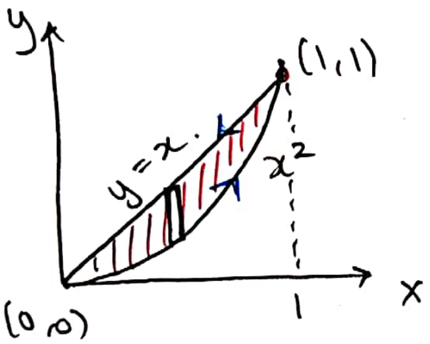
We know two objects of mass m_1 and m_2 exert a mutual attractive force that is proportional to

$$\sim \frac{G m_1 m_2}{r^2}$$

Why is sufficient to assume the force on P is due to the entire mass concentrated at O.

(To address later)

Example : Verify Green's Thm (Stokes version) for $\underline{F} = (xy + y^2, x^2)$.



Where D. is shown left.

i.e. verify :

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \underline{F} \cdot d\underline{r}$$

① $$LHS = \iint_D \left[2x - (x + 2y) \right] \cdot dA$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y) \cdot dy \cdot dx$$

$$= \int_{x=0}^{1} \left[x(x - x^2) - (x^2 - x^4) \right] \cdot dx$$

$$= \int_0^1 (-x^3 + x^4) \cdot dx = -\frac{1}{4} + \frac{1}{5} = -\frac{1}{20}.$$

② Set $\underline{r}_1(t) = (t, t^2)$, $\quad 0 \leq t \leq 1$.

$\underline{r}_2(t) = (t, t)$, $\quad t = 1 \ldots 0$.

$$\text{RHS} = \oint_C \underline{F} \cdot d\underline{r}$$

$$= \int_0^1 (t^3 + t^4, t^2) \cdot \underbrace{(1, 2t)}_{\underline{r}_1'(t)} \cdot dt$$

$$+ \int_1^0 (t^2 + t^2, t^2) \cdot \underbrace{(1, 1)}_{\underline{r}_2'(t)} \cdot dt$$

$$= \int_0^1 [t^3 + t^4 + 2t^3] \cdot dt + \int_1^0 (2t^2 + t^2) \cdot dt$$

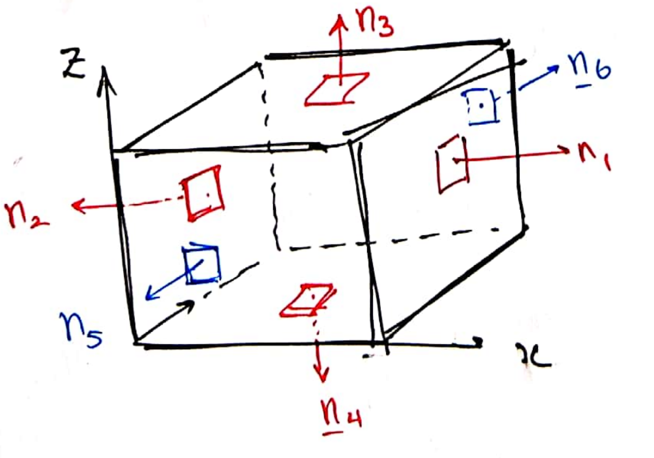
$$= \left(\frac{3}{4} + \frac{1}{5}\right) + t^3 \Big|_1^0$$

$$= \frac{19}{20} - 1 = -\frac{1}{20}. = \text{LHS} \quad ✓$$

Example: Verify the div. thm. for

$$\underline{F} = (2x - z, x^2 y, -xz^2)$$

and volume V = unit cube $[0,1]^3$.



Check:

$$\iiint_V \nabla \cdot \underline{F} \, dV = \iint_S \underline{F} \cdot \underline{n} \, dS.$$

$$LHS = \iiint_V (2 + x^2 - 2xz) \cdot dV.$$

$$= \frac{11}{6}.$$

To check RHS:

$(x=1)$ $\underline{n}_1 = (1,0,0)$ $\underline{F} \cdot \underline{n}_1 = (2-z, y, -z^2) \cdot (1,0,0)$
$= 2-z.$

$(x=0)$ $\underline{n}_2 = (-1,0,0)$, $\underline{F} \cdot \underline{n}_2 = (-z, 0, 0) \cdot (-1,0,0)$
$= z$

$(z=1)$ $\underline{n}_3 = (0,0,1)$, $\underline{F} \cdot \underline{n}_3 = (2x-1, x^2 y, -x) \cdot (0,0,1)$
$= -x.$

... continue this yourself.

Check integrals along surfaces S_1, S_2

$$I_{12} = \left(\iint_{S_1} + \iint_{S_2} \right) \underline{F} \cdot \underline{n} \, dS = \iint_{S_1} (2-z) \, dS + \iint_{S_2} z \, dS.$$

For S_1, S_2 $0 \le y \le 1$ $dS = dy \, dz$
$0 \le z \le 1.$

$$I_{12} = \iint_{S_1} 2 \, dS = 2 \cdot \text{area} = 2.$$

verify $\iint_S \underline{F} \cdot \underline{n} \, dS = \frac{11}{6}.$

Getting Started PS 3.

Q3. (b). Show.

$$(*) \qquad \iiint_{\Omega} \phi \nabla\cdot \underline{F}\, dV = \iint_{\partial\Omega} \phi E \cdot d\underline{S} - \iiint_{\Omega} E\cdot(\nabla\phi)\, dV.$$

(ϕ = scalar func. of x, y, z).

Suggests to try div. thm on $\phi \underline{E}$

$$\Rightarrow \quad \iiint_{\Omega} \nabla\cdot(\phi \underline{E})\, dV = \iint_{\partial\Omega} \phi \underline{E} \cdot \underline{n}\, dS. \quad , (**)$$

Expand $\nabla\cdot(\phi \underline{E}) = \nabla\phi \cdot \underline{F} + \phi(\nabla\cdot\underline{E})$

Subst. and done.

(c) Show:

$$\iiint_{\Omega} u \nabla^2 v\, dV = \iint_{\partial\Omega} u \frac{\partial v}{\partial n}\, dS - \iiint_{\Omega} \nabla u \cdot \nabla v\, dV$$

Use (*) set $\phi = u$, and want $\nabla\cdot\underline{F} = \nabla^2 v$

but $\nabla^2 v = \nabla\cdot(\nabla v) \Rightarrow$ set $\underline{F} = \nabla v$