LECTURE 13.

STOKES' THEOREM

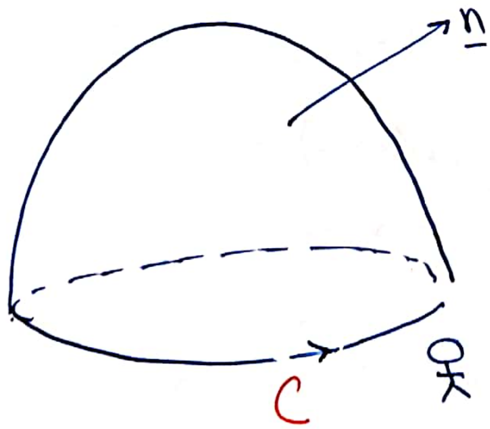
Recall:
$$\oint_C \underline{F} \cdot d\underline{r} = \iint_D \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)}_{(\nabla \times \underline{F}) \cdot \underline{k}} dA$$

THM 9.2 : (STOKES' THM)

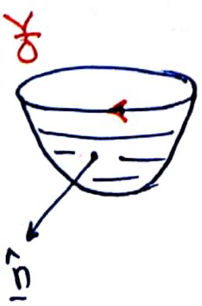
Let S be a *smooth, bounded, open surface with a boundary curve C. Let $\underline{n}$ be the outward normal and C is correspondingly oriented with the surface. Then

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_S (\nabla \times \underline{F}) \cdot \underbrace{d\underline{S}}_{\underline{n}\, dS}$$

* we don't need smoothness.



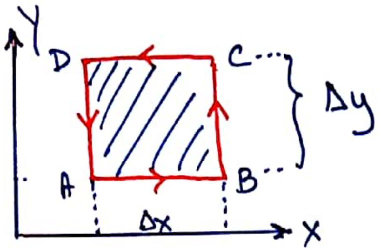
Correspondingly oriented means if you walk around C with the surface on your left, then $\underline{n}$ is in the direction of your head.



* The next few pages were covered quickly and pre-written so examine video recordings.

We want to explain why GREEN'S THM makes sense.

THM: Let C be the contour around a small box ABCD



Then $\oint_C \underline{F} \cdot d\underline{r} \approx (\nabla \times \underline{F}) \cdot \underline{k} \; \Delta x \, \Delta y$.

So the curl, $\nabla \times \underline{F}$, is a measure of how much the field "spins" (per unit area).

PROOF. Let $\underline{F} = (F_1, F_2, F_3)$. For a generic function $G(x,y)$ [ignore z], using Taylor expansions,

$$G(A) = G(x,y)$$
$$G(B) = G(x+\Delta x, y) \approx G(A) + \frac{\partial G}{\partial x}\Delta x$$
$$G(C) = G(x+\Delta x, y+\Delta y) \approx G(A) + \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y$$
$$G(D) = G(x, y+\Delta y) \approx G(A) + \frac{\partial G}{\partial y}\Delta y$$

As long as $\Delta x, \Delta y$ small,

$$\int_A^B \underline{F} \cdot d\underline{r} \approx F_1(A)\,\Delta x$$

$$\int_C^D \underline{F} \cdot d\underline{r} \approx -\left[F_1(A) + \frac{\partial F_1}{\partial y}\Delta y\right]\Delta x$$

So $\left(\int_A^B + \int_C^D\right) \underline{F} \cdot d\underline{r} \approx -\frac{\partial f_1}{\partial y} \Delta x \Delta y.$

Similarly $\left(\int_B^C + \int_D^A\right) \underline{F} \cdot d\underline{r} \approx \frac{\partial F_2}{\partial x} \Delta x \Delta y.$

So the sum,

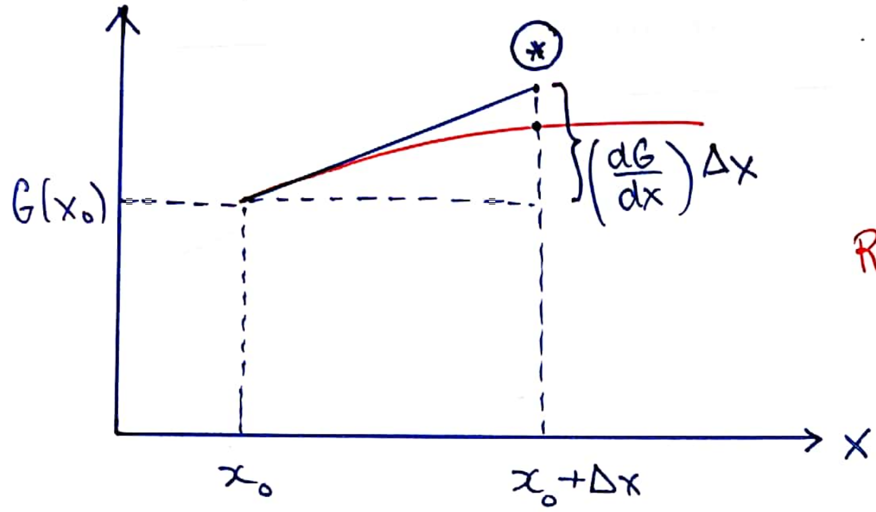
$$\oint_C \underline{F} \cdot d\underline{r} \approx \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \Delta x \Delta y$$

which can be written as,

$$\oint_C \underline{F} \cdot d\underline{r} \approx (\nabla \times \underline{F}) \cdot \underline{K} \; \Delta x \Delta y.$$

$\square$

Taylor expansion (if you didn't learn it)

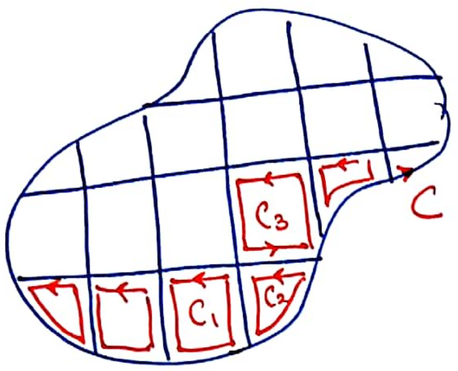


Red curve = $G(x)$.

The point ⊛ approximates $G(x+\Delta x)$.

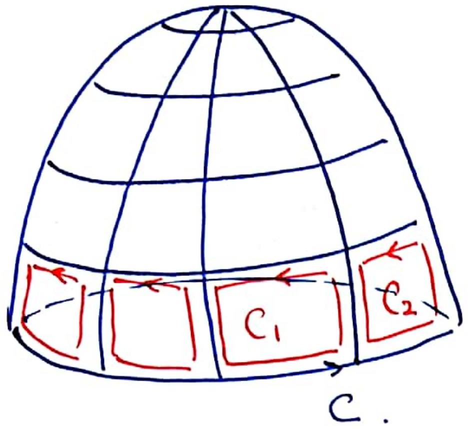
So $G(x_0+\Delta x) \approx G(x_0) + \left(\frac{dG}{dx}\right)\Delta x$.

Green's Theorem is then based on that idea:



Split the area S with bounding curve C into smaller boxes with curves C_i.

By the picture,
$$\oint_C \underline{F} \cdot d\underline{r} = \sum_{i=1}^{N} \int_{C_i} \underline{F} \cdot d\underline{r} \qquad N \to \infty$$
$$\approx \sum_i (\nabla \times \underline{F}) \cdot \underline{k} \, \Delta x \Delta y$$
$$= \iint_S (\nabla \times \underline{F}) \cdot \underline{k} \, dx\, dy$$



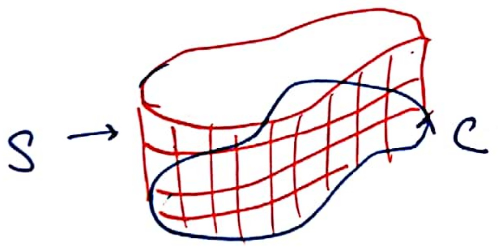
$$\int_C \underline{F} \cdot d\underline{r} = \sum_{i=1}^{N} \int_{C_i} \underline{F} \cdot d\underline{r}$$

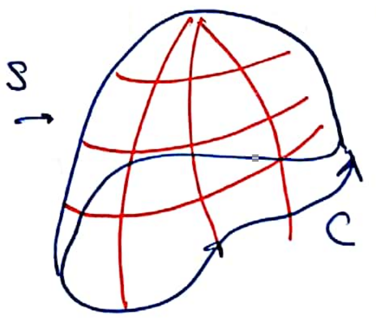
$$= \sum_{i=1}^{N} (\nabla \times \underline{F}) \cdot \underline{n} \; dS.$$

$$= \iint_S (\nabla \times \underline{F}) \cdot \underline{n} \; dS.$$

[This was the end of the pre-written material]

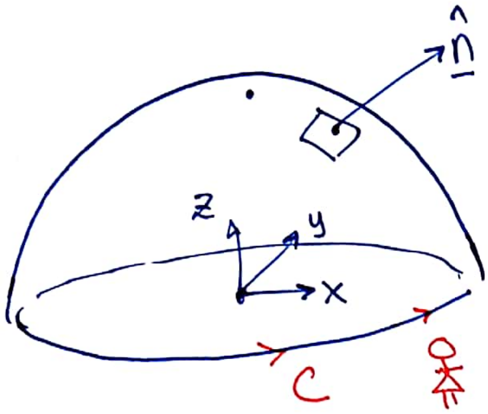
Notice that Stokes' Thm states $\oint_C \underline{F} \cdot d\underline{r}$ is the same value as long as S in $\iint_S (\nabla \times \underline{F}) \cdot \underline{n} \; dS$ has C as its bounding curve.





<u>Example</u>: Verify Stokes' Thm for $\underline{F} = (y, -x, z)$, S is the upper hemisphere with normal as in figure

Verify $\iint_S (\nabla \times \underline{F}) \cdot \underline{n} \; dS = \oint_C \underline{F} \cdot d\underline{r}$



$$RHS = \oint_C \underline{F} \cdot d\underline{r} \quad , \text{where } C: \underline{r}(t) = (\cos t, \sin t, 0)$$
$$0 \le t \le 2\pi .$$

$$= \int_{t=0}^{2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0) \, dt$$

$$= -2\pi$$

$$LHS = \iint_S \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & -x & z \end{vmatrix} \cdot \underbrace{\underline{n} \, dS}_{(\underline{r}_u \times \underline{r}_v)\, du \cdot dv} .$$

$$= \iint_S (0, 0, -2) \cdot \underline{n} \, dS$$

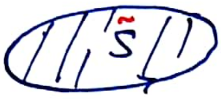
$$\underline{n} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \iint_S \left(\frac{-2z}{1} \right) dS$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\frac{\pi}{2}} (-2\cos\varphi) \, (1^2 \sin\varphi) \cdot d\varphi \cdot d\theta$$

$$= -2\pi = RHS \quad \checkmark$$

We can choose any surface with C as bounding curve. Try the planar disc?



C

Try $\iint_{\tilde{S}} (\nabla \times \underline{F}) \cdot \underline{n} \, dS$

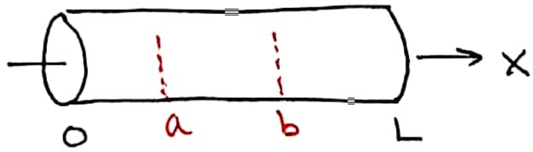
$$= \iint_{\tilde{S}} (0,0,-2) \cdot (0,0,1) \, dS .$$

$$= -2 \cdot \iint_{\tilde{S}} dS = -2\pi .$$

THE HEAT EQUATION.

LECTURE 14.

Consider a bar $x \in [0, L]$, with vertical sides insulated.



- Let $T(x,t)$ be temperature.
- The energy density is $\rho c T$

ρ → density $(\mathrm{kg/m})$; c → specific heat $(\mathrm{J/kg \cdot K})$; T → temperature (K)

- By energy conservation,

$$\underbrace{\frac{d}{dt}\int_a^b \rho c T \cdot dx}_{\text{change in internal energy}} = \underbrace{q(a,t) - q(b,t)}_{\text{flow in at } x=a \text{ − flow out at } x=b.}$$

- q is the heat flux at x and time t where $q > 0$ is flow in the $+$ direction.

- $$\frac{d}{dt}\int_a^b \rho c T \cdot dx = -\int_a^b \frac{\partial q}{\partial x} \cdot dx$$

$$\Rightarrow \int_a^b \rho c \frac{\partial T}{\partial t} \cdot dx = -\int_a^b \frac{\partial q}{\partial x} \cdot dx$$

$$\Rightarrow \int_a^b \left(\rho c \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x}\right) \cdot dx = 0.$$

Since a, b are arbitrary, (Bump Lemma)

$$\boxed{\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}}$$

By Fourier's Law,

$$q(x,t) = -K \frac{\partial T}{\partial x}$$

thermal conductivity.

$$\Rightarrow \quad \rho c \frac{\partial T}{\partial t} = K \cdot \frac{\partial^2 T}{\partial x^2}.$$

HEAT EQN. 1D.

$$\boxed{\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}} \quad \text{where } \kappa = \frac{K}{\rho c}.$$

HEAT EQN 3D.

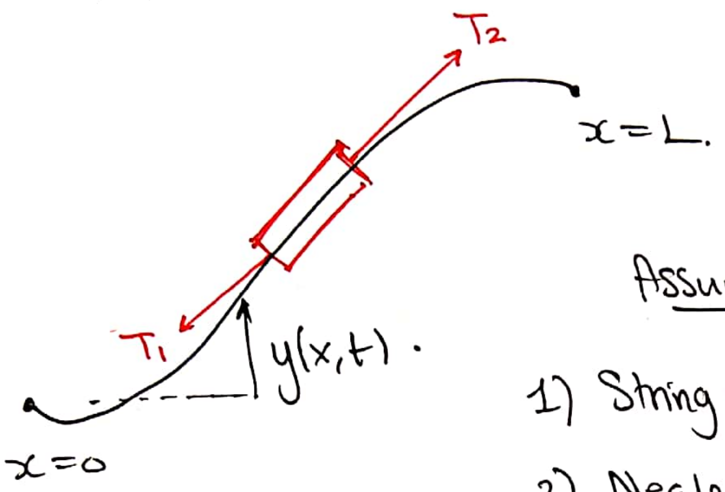
$$\boxed{\frac{\partial T}{\partial t} = \kappa \cdot \nabla^2 T = \kappa \left(T_{xx} + T_{yy} + T_{zz} \right).}$$

Two types of boundary conditions:

Fix temperature (Dirichlet) : set $T(\underline{x},t)$ on the boundary.

Fix flux (Neumann): set $q(\underline{x},t)$ on the boundary.

WAVE EQN

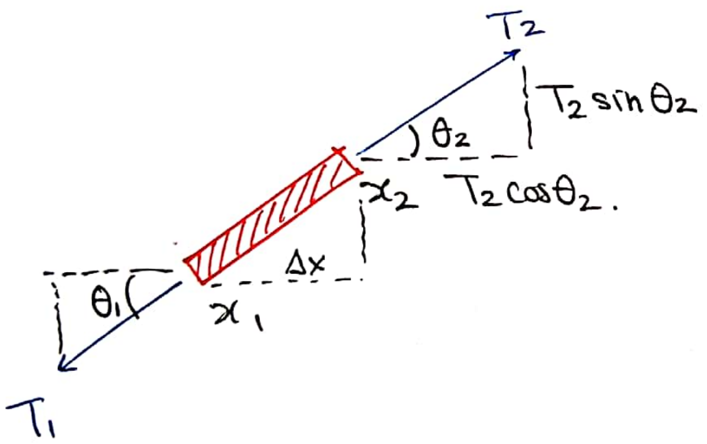


<u>Assumptions</u> :

1) String tension T, density ρ.
2) Neglect gravity + air resist.
3) Small deflections $\left|\frac{\partial y}{\partial x}\right| \ll 1$.

small

Apply Newton's Laws to the segment
$[x_1, x_2]$, $x_2 = x_1 + \Delta x$, $\Delta x \ll 1$.



Balance forces: $(F = ma)$.

<u>Horz</u>: $T_2 \cos\theta_2 = T_1 \cos\theta_1$ (1)

<u>Vert</u>: $(\rho \Delta x) y_{tt}$ = sum of forces.

where $y_{tt} = y_{tt}(x_0, t)$
where $x_0 \in [x_1, x_2]$.

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On vertical,

$$\underbrace{(\rho \Delta x)}_{*} y_{tt}(x_0, t) = T_2 \sin\theta_2 - T_1 \sin\theta_1 \qquad (2)$$

$*$ note $\rho \Delta s = \rho \sqrt{\Delta x^2 + \Delta y^2} = \rho \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \approx \rho \Delta x$

note also $\sin\theta \sim \tan\theta \sim \frac{\partial y}{\partial x}$ if θ is small (since $\cos\theta \approx 1$)

By (1), since θ_1 and θ_2 small $\cos\theta_1 \approx 1$

and $\cos\theta_2 \approx 1$

$\therefore$ $T_1 \approx T_2 \equiv T$ (Tension is constant).

By (2) $\Rightarrow (\rho\,\Delta x)\, y_{tt}(x_0,t) = T \cdot \left\{ \frac{\partial y}{\partial x}(x_2,t) - \frac{\partial y}{\partial x}(x_1,t) \right\}.$

$$\Rightarrow \rho\, y_{tt}(x_0,t) = T \cdot \left\{ \frac{y_x(x_2,t) - y_x(x_1,t)}{\Delta x} \right\}$$

By MVT, the RHS $= T \cdot y_{xx}(a,t)$ $a \in [x_1, x_2]$.

$\therefore$ In limit $\Delta x \to 0 \Rightarrow y_{tt} = y_{xx} \cdot \frac{T}{\rho}$.

WAVE EQUATION 1D :

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad c = \sqrt{\frac{T}{\rho}}}$$

This can be generalised. In 3D,

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \nabla^2 y = c^2 \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} \right)}$$

How to begin solving such equations?

Consider

u = heat in 1D bar.

$$\begin{cases} u_t = k u_{xx} & x \in [0,L],\ t > 0. \\ u(0,t) = 0 & \\ u(L,t) = 0. & \Big\}\ \text{Boundary conditions, } t > 0. \\ u(x,0) = f(x) & \text{Initial condition, } x \in [0,L]. \end{cases}$$

On a leap of faith, we assume,

$$u(x,t) = X(x) \cdot G(t).$$

i.e. solution is separable.

$\underline{\text{PDE}} \Rightarrow \quad XG' = K\,G \cdot X''$

$$\Rightarrow \underbrace{\frac{1}{K}\frac{G'(t)}{G(t)}}_{\text{func. of } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{func. of } x} = \text{constant}$$

Assume const $= -\lambda^2 < 0$.

Solve $\begin{cases} G' = -K\lambda^2 G & (1) \\ X'' + \lambda^2 X = 0. & (2) \end{cases}$

(1) $\Rightarrow$ $\boxed{G(t) = D \cdot e^{-K\lambda^2 t}}$, D constant. (3)

(2) $\Rightarrow$ subst. e.g. $X = e^{rx}$, ... solve r.

$\Rightarrow$ $\boxed{X(x) = A\cos(\lambda x) + B\sin(\lambda x)}$ (4)

What are A, B, D ?

Since $u(0,t) = 0 \Rightarrow X(0) \cdot G(t) = 0.$

$\Rightarrow \quad X(0) = 0.$ ($G(t)$ non-triv.)

$u(L,t) = 0 \Rightarrow X(L) \cdot G(t) = 0$

$\Rightarrow \quad X(L) = 0.$

Subst. into (4) $\Rightarrow \begin{cases} X(0) = 0 \Rightarrow A = 0. \\ X(L) = 0 \Rightarrow B \cdot \sin(\lambda L) = 0. \\ \quad \therefore \lambda L = n\pi, \quad n \in \mathbb{Z} \end{cases}$

$\therefore$ We have shown possible solutions:

$$u_n(x,t) = D_n \cdot e^{-\lambda_n^2 kt} \cdot B_n \sin(\lambda_n \cdot x)$$

where $\lambda_n = \frac{n\pi}{L}$, , $n \in \mathbb{Z}$.

Simplify to :

$$\boxed{u_n(x,t) = \hat{B}_n \cdot e^{-\lambda_n^2 kt} \sin(\lambda_n x). \quad \text{where } \lambda_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \ldots \in \mathbb{Z}^+.}$$

The question is: how to impose $u(x,0) = f(x)$.

By linearity, we can add solutions to obtain more solutions. So we claim that a general solution is.

$$u(x,t) = \sum_{n=0}^{\infty} \hat{B}_n e^{-k\lambda_n^2 t} \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{L}.$$

The I.C. tells us

$$u(x,0) = \boxed{f(x) = \sum_{n=0}^{\infty} \hat{B}_n \sin\left(\frac{n\pi x}{L}\right)}.$$

How do we solve for $\hat{B}_n$? This is theory of Fourier Series.