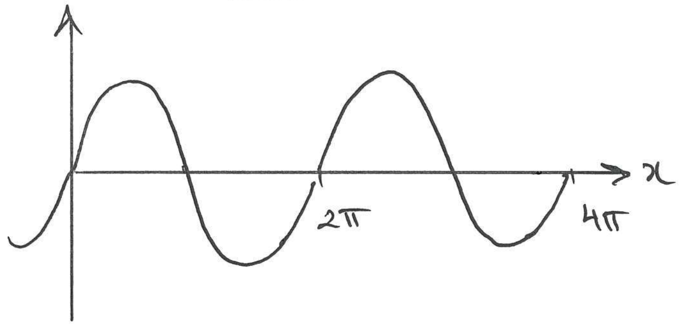
PROBLEM SET 6.

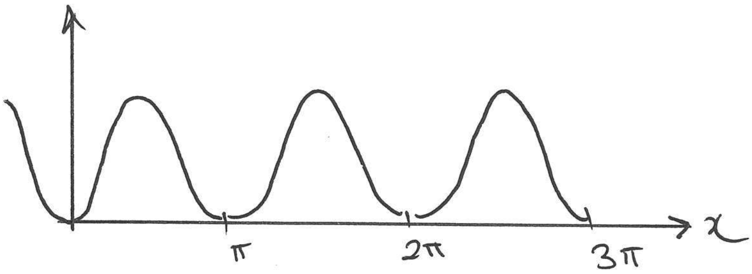
1. (a)



$\sin x \rightarrow$ fundamental period is 2π

odd

(b)



even

note $\sin^2 x = \frac{1}{2} \cdot \{1 - \cos 2x\}$

fundamental period $= \pi$

* note that a function $\cos(nx)$ or $\sin(nx)$ has fundamental period $\frac{2\pi}{n}$.

(c) $f(x) = \sin x \cos 3x$.

Use $\sin nx \cdot \cos mx = \frac{1}{2}\left\{\sin((n-m)x) + \sin((n+m)x)\right\}$

so $f(x) = \frac{1}{2}\left\{\underbrace{\sin(-2x)}_{\text{period } \pi} + \underbrace{\sin(4x)}_{\text{period } \frac{\pi}{2}}\right\}$

$\therefore$ fundamental period is π

Also $\sin x$ = odd & $\cos 3x$ = even

$\Rightarrow$ $f(x)$ is odd.

(d) $f(x) = \sin x + \sin(\sqrt{2}x)$

period 2π period $\frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$

so 2π and $\sqrt{2}\pi$ have no common factor as we need $m\cdot(2\pi) = n\cdot(\sqrt{2}\pi)$ for integers m, n.

Note $\sin x$ odd & $\sin(\sqrt{2}x)$ odd $\Rightarrow f(x)$ is odd.

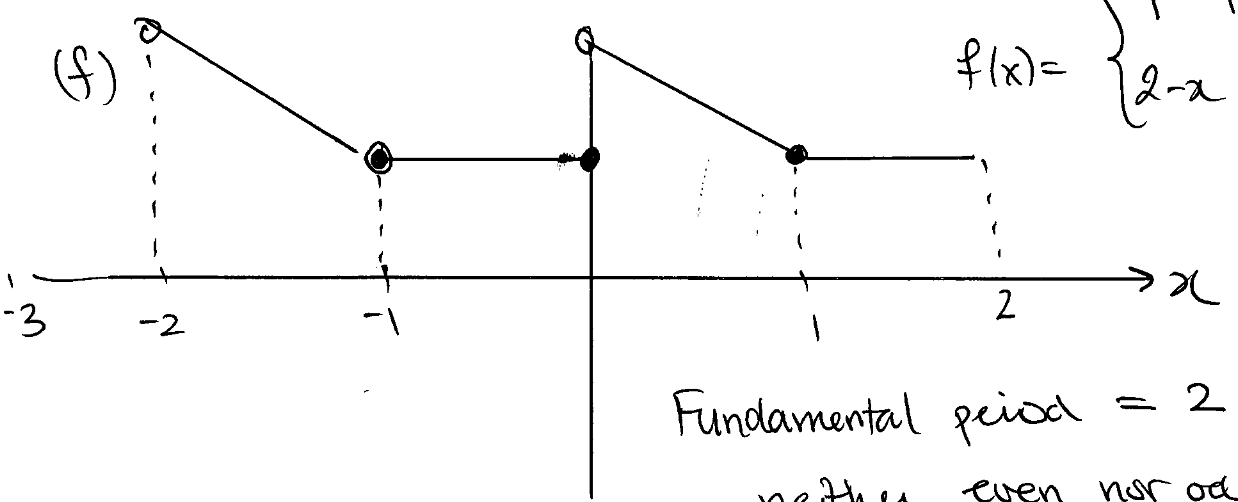
(e) $f(x) = \sin x \underline{\exp(-\cos(x^2))}$

not a periodic function since exp. not periodic.

Check even or odd:

$$\begin{aligned} f(-x) &= \sin(-x)\exp(-\cos(-x)^2) \\ &= -\sin(x)\exp(-\cos x^2) \\ &= -f(x) \end{aligned}$$

so f is odd.

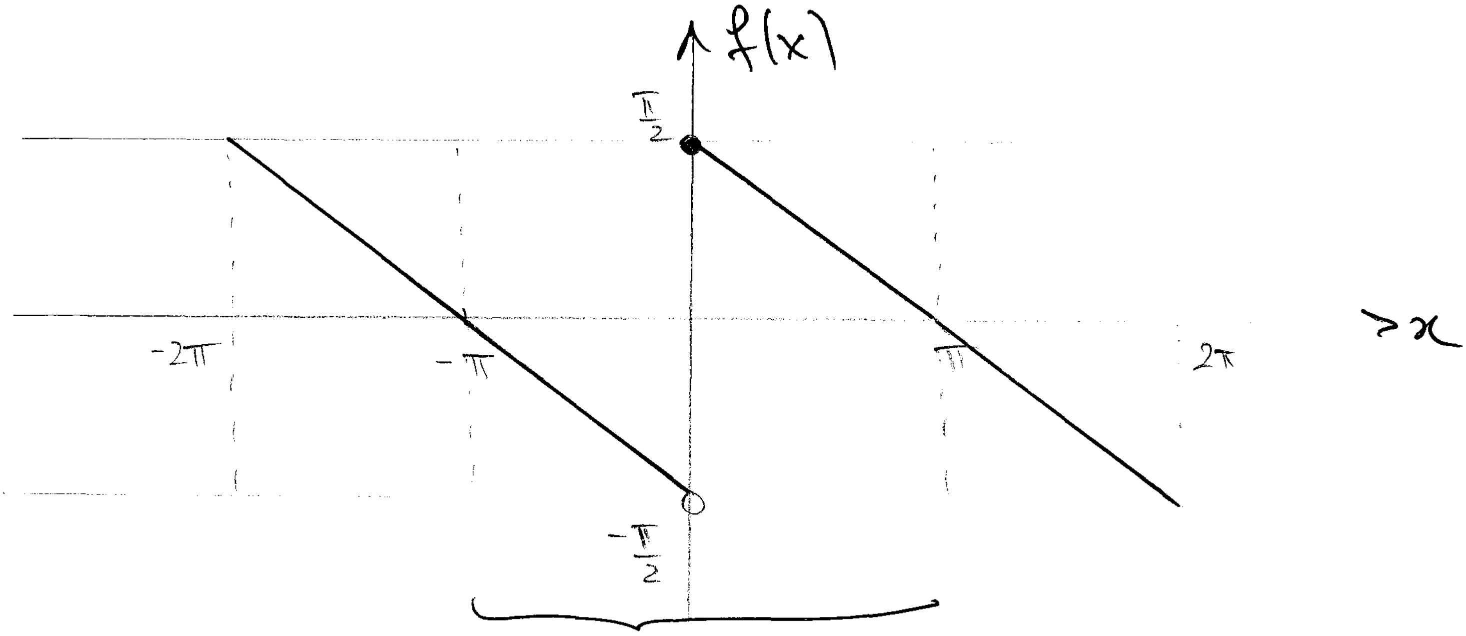
(f) 

$f(x) = \begin{cases} 1 & \text{if } x \in (-1, 0] \\ 2-x & \text{if } x \in (0, 1] \end{cases}$

Fundamental period $= 2$

neither even nor odd

#2. $f(x) = \begin{cases} \frac{1}{2}(-\pi - x) & x \in (-\pi, 0]. \\ \frac{1}{2}(\pi - x) & x \in (0, \pi]. \end{cases}$



main window $(-\pi, \pi]$

2π-periodic.

* Seek $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$

note firstly that $f(x)$ is odd about $x=0$

since $f(-x) = -f(x)$.

Thus we know that $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \cos nx \cdot dx}_{\text{odd}}$

$= 0 \quad \forall n \geq 0.$

So that leaves

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \cdot dx$$

(double up the integration due to $(-\pi, 0)$)

Thus

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\pi - x)\sin nx \cdot dx$$

Integration by parts: $u = \pi - x$ $\quad dv = \sin nx$

$du = -dx$ $\quad v = \frac{-1}{n}\cos nx$

$$\therefore b_n = \frac{1}{\pi}\left\{ (\pi - x)\left(\frac{-1}{n}\cos nx\right)\Bigg|_0^{\pi} - \underbrace{\int_0^{\pi} \frac{1}{n}\cos nx \cdot dx}_{=0.} \right\}$$

$$= \frac{1}{\pi}\frac{1}{n}\pi \cdot \cos(n \cdot 0)$$

$\boxed{b_n = \frac{1}{n}}$ for $n \geq 1$

$$\therefore \boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{1}{n}\sin nx}$$

(c) Assuming series converges, then notice $f\left(\frac{\pi}{2}\right)$ is $\frac{1}{2}(\pi - x)\Big|_{x=\pi/2} = \frac{\pi}{4}$. Thus,

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n}.$$

But $\sin\left(\frac{\pi n}{2}\right) = 0$ if n is even

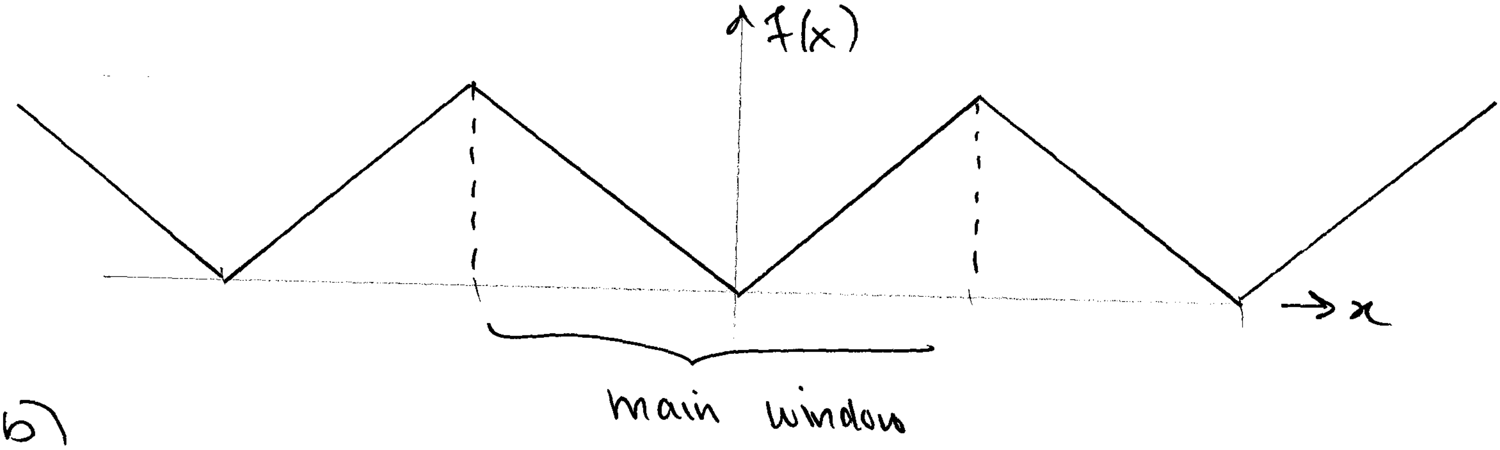
When n is odd, let $n = 2k+1$, for $k = 0, 1, 2, \ldots$

Thus $\sin\left(\frac{\pi}{2}(2k+1)\right) = (-1)^k$.

Thus $\frac{\pi}{4} = \sum_{k=0}^{\infty} (-1)^k/(2k+1) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

#3.

(a) $f(x) = \begin{cases} -x & x \in (-\pi, 0] \\ x & x \in (0, \pi] \end{cases}$, $f(x+2\pi) = f(x)$



(b)

* Note that $f(x)$ is even since $f(-x) = f(x)$.

* We seek $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$

Since f even, we expect $b_n = 0 \ \forall n$. Is this true?

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underset{\text{even}}{f(x)} \ \underset{\text{odd}}{\sin nx}}_{\text{odd}} \cdot dx = 0, \ n \geq 1$$

Thus we need $a_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \cos nx \cdot dx$, $n \geq 0$

Firstly, do $n=0$ case separately:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \cdot dx = \frac{1}{\pi}\cdot\pi^2 = \pi.$$

Next for $n \geqslant 1$,

$$a_n = \frac{2}{\pi}\cdot\int_0^{\pi} x\cdot\cos(nx)\cdot dx$$

Int. by parts

$$= \frac{2}{\pi}\left\{ \frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\cdot\int_0^{\pi}\sin nx\cdot dx.\right\}$$

$$= \frac{2}{\pi}\left\{ \underbrace{\frac{\pi}{n}\sin(n\pi)}_{=0} + \frac{1}{n^2}\cos nx\Big|_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\cdot\frac{1}{n^2}\left\{\cos(\pi n) - 1\right\}$$

But $\cos(n\pi) = (-1)^n$

$$\therefore\quad a_n = \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\}$$

Note that if n is even then all $a_n = 0$.

So let $n = 2k+1$, $k = 0, 1, 2, \ldots$

$$a_{2k+1} = \frac{2}{\pi(2k+1)^2}\cdot(-2) = \frac{-4}{\pi(2k+1)^2}.$$

Thus $f(x) \sim \frac{\pi}{2} + \left(-\frac{4}{\pi}\right) \cdot \sum_{k=0}^{\infty} \frac{\cos[(2k+1)x]}{(2k+1)^2}$

(c) Assume series converges at all x.

* Try $x = \pi$. Then,

$$\cos[(2k+1)\pi] = -1$$

* But looking at $f(x)$, we see $f(\pi) = $ pi.

* Thus,

$$\text{pi} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{0}^{\infty} \frac{(-1)}{(2k+1)^2}$$

$$\therefore \sum_{0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

#4. This question is just bookwork. In lectures, we showed, e.g.

$$\int_{-\pi}^{\pi} \sin(nx)\cdot\sin(mx)\cdot dx = \pi\cdot\delta_{mn}.$$

Easiest way is just to do substitution. For instance,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\cdot dx. = I.$$

Let $\frac{n\pi x}{L} = n.s \Rightarrow x = \frac{L.s}{\pi}$

Then $dx = \frac{L}{\pi}.ds$ and $\begin{cases} x=-L \Rightarrow s = -\pi \\ x = L \Rightarrow s = \pi \end{cases}$

Thus $I = \int_{-\pi}^{\pi} \sin(ns)\cdot\sin(ms)\left(\frac{L}{\pi}\right) ds$

$$= \frac{L}{\pi}(\pi\cdot\delta_{mn})$$

$$= L\,\delta_{mn}.$$

all the other results are similar

* Be careful to treat the $n=m=0$ case separately.

$\langle \sin(0x), \sin 0x\rangle = \int_{-L}^{L} 0.0.dx = 0.$

$\langle \sin(0x), \cos(0x)\rangle = \int_{-L}^{L} 0.\cos(0).dx = 0$

$\langle \cos(0x), \cos(0x)\rangle = \int_{-L}^{L} (1).(1).dx$

$= 2L$

(b). Let $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\}$.

Again we can scale things ... but let's do directly. Multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate:

$$\int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \cdot dx = \frac{A_0}{2} \cdot \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cdot dx$$

$$+ \sum_{n=1}^{\infty} \left\{ A_n \cdot \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \right.$$

$$\left. + B_n \cdot \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \right\}$$

every $=0$ except $n=m$

$$\Rightarrow \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \cdot dx = B_m (L)$$

$$\therefore B_m = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \cdot dx, \quad m > 1$$

Exactly same for A_n coefs $\Rightarrow$ $A_m = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx$

$m \geq 0$.