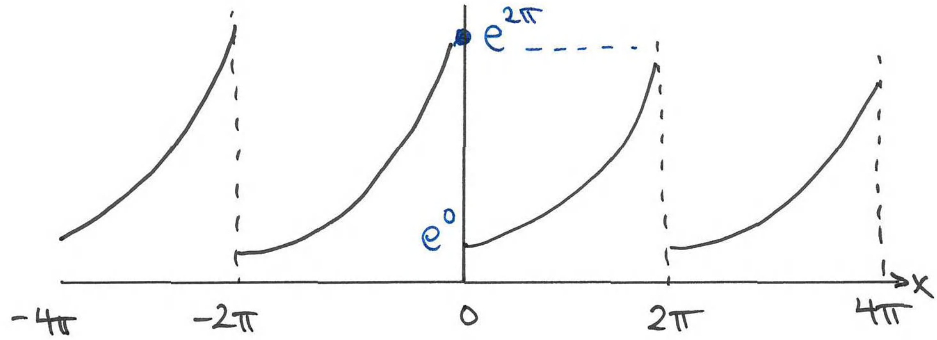
PROBLEM SET 7.

#1.

a)



As shown in Example 11.11, the 2π-periodic extension of $f(x) = e^x$ on $(0, 2\pi)$ is,

$$f(x) \sim \frac{(e^{2\pi}-1)}{2\pi} + \frac{(e^{2\pi}-1)}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{\cos(nx)}{1+n^2} - \frac{\sin(nx)}{1+n^2} \right\}$$

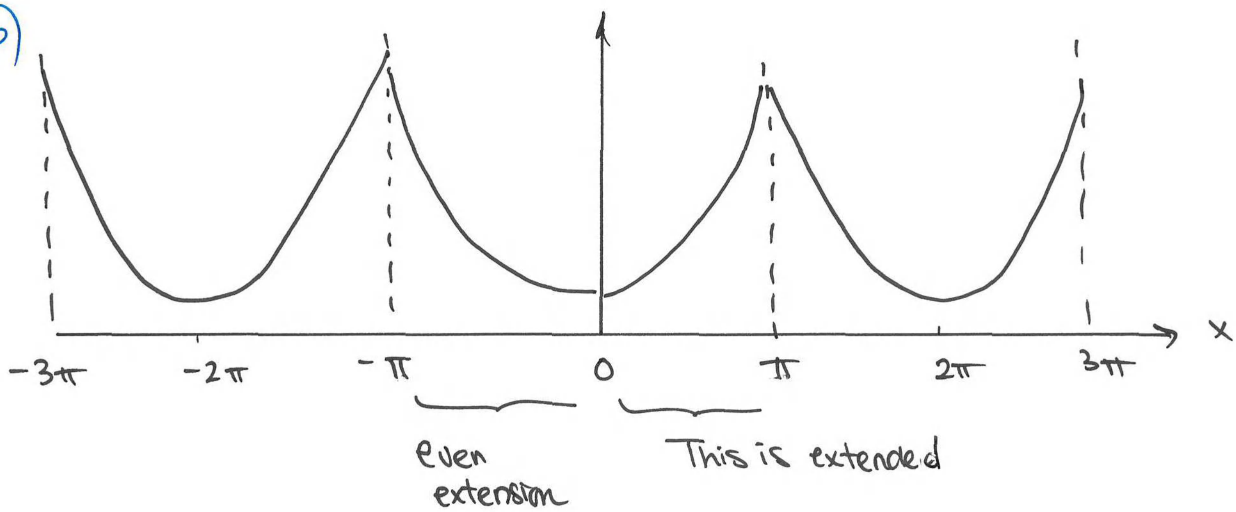
By Fourier convergence theorem,

$$\frac{f(0^-) + f(0^+)}{2} = \text{series at } x=0$$

$$\Rightarrow \frac{\frac{1}{2}(e^{2\pi}+1)\cdot\pi}{e^{2\pi}-1} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2}\cdot\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right) - \frac{1}{2}$$

b)



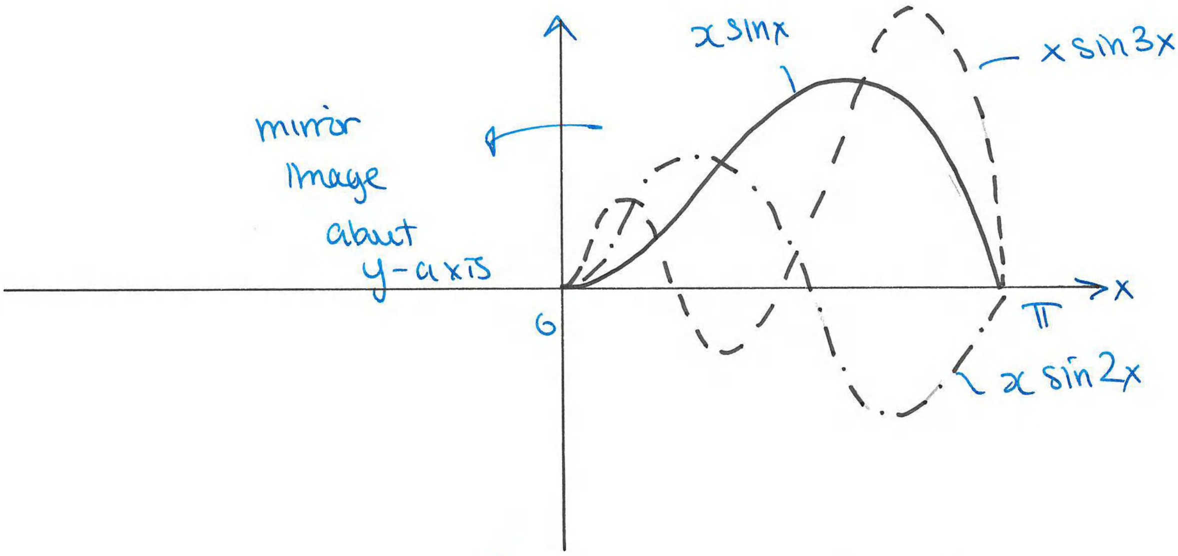
Now the even extension is continuous at $x=0$, so

$$e^0 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\cdot 0)$$

$$1 = \frac{1}{2}\cdot\frac{2}{\pi}(e^{\pi}-1) + \sum_{n=1}^{\infty} \frac{2}{\pi(1+n^2)}\left\{(-1)^n e^{\pi}-1\right\}.$$

#2. Define $f(x) = \underbrace{x}_{\text{odd}}\,\underbrace{\sin(px)}_{\text{odd}}$, $x \in (-\pi, \pi)$, p integer

So odd · odd = even $\Rightarrow$ $f(x)$ is even.



Write $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos nx + b_n \sin nx\right\}$

* note $f(x)$ is even $\Rightarrow b_n = 0 \ \forall n$.

Then, $a_n = \frac{2}{\pi}\cdot\int_0^{\pi} x\cdot\sin(px)\cdot\cos(nx)\cdot dx$

$$= \frac{1}{\pi}\cdot\int_0^{\pi} x\left\{\sin((p-n)x) + \sin((p+n)x)\right\}\cdot dx$$

using §10.2.1 identity.

Now note $\displaystyle\int_0^{\pi} x\cdot\sin(qx)\cdot dx = -\frac{x}{q}\cdot\cos qx\Big|_0^{\pi} + \frac{1}{q}\int_0^{\pi}\cos qx\cdot dx$

The second integral $= 0$ if q is integer.

Thus, $\displaystyle\int_0^{\pi} x\sin(qx)\cdot dx = \frac{-\pi}{q}\cos(q\pi) = \frac{-\pi(-1)^q}{q},$

Thus, we have that,

$$a_n = \frac{1}{\pi}\cdot(\pi)\cdot\left\{\frac{(-1)^{p-n+1}}{p-n} + \frac{(-1)^{p+n+1}}{p+n}\right\} \quad \text{if } p\neq n \text{ and } n\neq 0.$$

Factor out $(-1)^{n+p+1}$

$$\Rightarrow \quad a_n = (-1)^{n+p+1}\left\{\frac{(-1)^{-2n}}{p-n} + \frac{1}{p+n}\right\}$$

But $(-1)^{-2n} = 1$ and $\displaystyle\frac{1}{p-n} + \frac{1}{p+n} = \frac{2p}{p^2-n^2}$

Thus $\boxed{a_n = \dfrac{2p(-1)^{n+p}}{n^2-p^2}}$, $p\neq n$, $n\neq 0$.

It remains to do case of $n=0$ and $n=p$.

Case $n=0$: $\displaystyle a_0 = \frac{2}{\pi}\cdot\int_0^{\pi} x\sin(px)\cdot dx$

$$= \frac{2}{\pi}\,\frac{\pi}{p}\,(-1)^{p+1}$$

$$\therefore \quad \boxed{a_0 = \frac{2}{p}(-1)^{p+1}}$$

(Case $n=p$):

We have $a_p = \frac{1}{\pi} \cdot \int_0^{\pi} x \left\{ \sin(0 \cdot x) + \sin(2px) \right\} dx$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin(2px) \cdot dx$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{2p} \cdot (-1)^{2p+1}$$

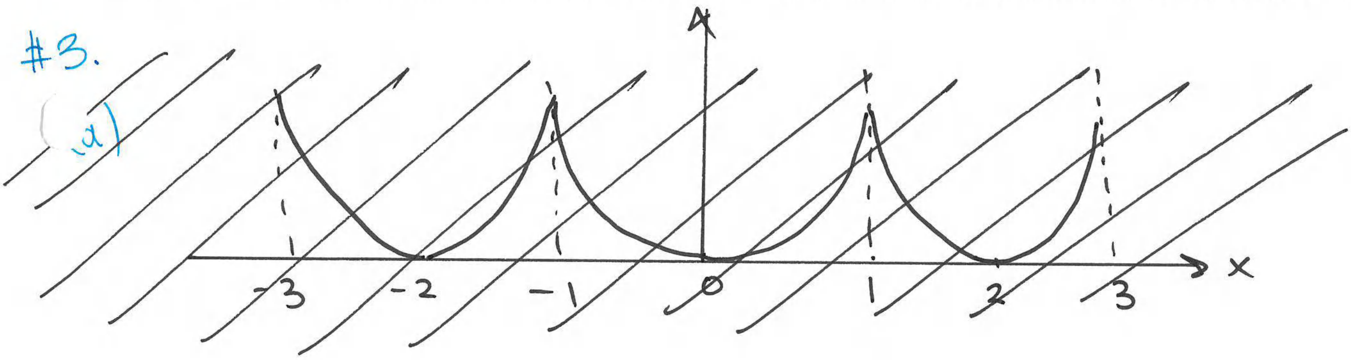
$$\boxed{a_p = \frac{-1}{2p}}$$

$$\therefore \boxed{f(x) \sim \frac{(-1)^{p+1}}{p} - \frac{1}{2p} \cdot \cos px + \sum_{\substack{n=1 \\ n \neq p}}^{\infty} \frac{2(-1)^{n+p} p}{n^2 - p^2} \cdot \cos(nx)}$$

(d) See separate sheet.

#3.

(a)



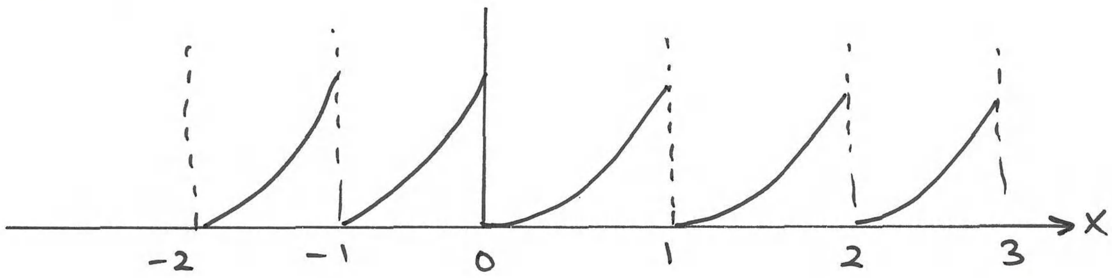
#2. c)

If $p=1, x=0 \Rightarrow 0 = \frac{(-1)^2}{1} - \frac{1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cdot 2 \cdot \Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} = \frac{1}{4}$

If $p=2, x=\pi \Rightarrow \pi \sin(2\pi) = \frac{(-1)^3}{2} - \frac{1}{4} \cos(2\pi) + \frac{4(-1)^3}{1-4} \cos\pi$

$$+ \sum_{n=3}^{\infty} \frac{4(-1)^{n+2}}{n^2-4} \cos(n\pi).$$

#3. Let $f(x) = x^2$, $x \in (0,1)$.



General series: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$

Here $L = \frac{1}{2}$ for the function from $[-L, L]$

or alternatively $[0, 2L]$. See Remark 1.8.

Thus, need a_n and b_n.

$$a_n = \frac{1}{(1/2)} \cdot \int_0^1 (x^2) \cos(2n\pi x) \cdot dx$$

Integrate by parts and use $\sin(2n\pi) = 0$.

Then, for $n \neq 0$.

$$a_n = 2 \cdot \left(\frac{2n\pi}{4(n\pi)^3}\right) \cdot \cos(2n\pi) = \frac{1}{(n\pi)^2} \cdot (-1)^{2n\pi} = \frac{1}{(n\pi)^2}.$$

We must do the case $n=0$ separately.

$$a_0 = 2 \cdot \int_0^1 x^2 \cdot dx = \frac{2}{3}.$$

Finally, for $n \geqslant 1$

$$b_n = 2 \cdot \int_0^1 x^2 \sin(2n\pi x) \cdot dx$$

$$= \frac{1}{2(n\pi)^3} \cdot \left\{ -1 + (1 - 2(n\pi)^2) \cdot \cos 2n\pi \right\}$$

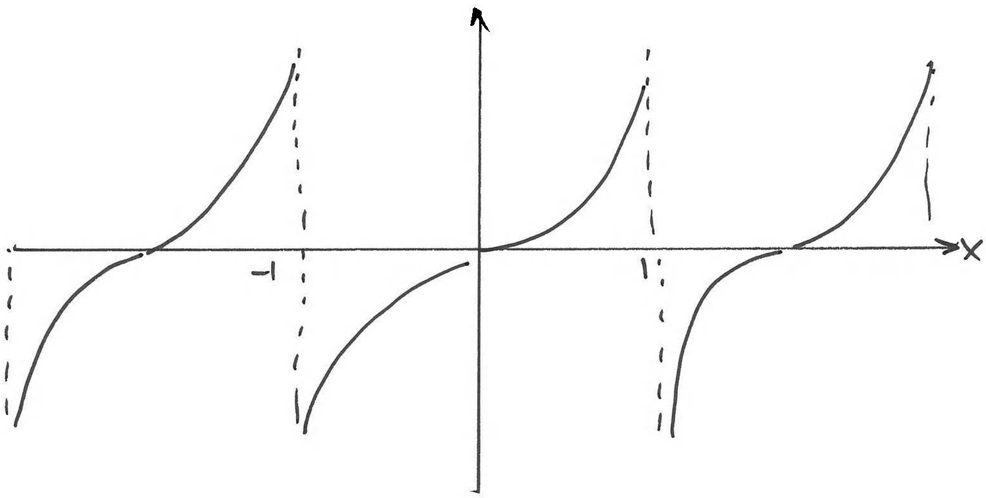
using $\sin(2n\pi) = 0$

Thus, $b_n = \dfrac{1}{2(n\pi)^3}(-2)(n\pi)^2$ using $\cos(2n\pi) = 1$

$$b_n = -\frac{1}{n\pi}, \quad n \geq 1.$$

$$\therefore f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left\{ \frac{1}{(n\pi)^2}\cos(2n\pi x) - \frac{1}{n\pi}\sin(2n\pi x) \right\}$$

(b)



So now $f(x)$ is odd $\Rightarrow$ Sine series with $a_n = 0 \ \forall n \geq 0$.

Now $b_n = \dfrac{2}{1}\displaystyle\int_0^{\cancel{1/2}\ 1} x^2 \sin(n\pi x) \cdot dx$ since $L = 1$

$$b_n = 2\int_0^{\cancel{1/2}\ 1} x^2 \sin(n\pi x) \cdot dx$$

$$= \frac{2}{(n\pi)^3} \cdot \left\{ -2 + \left(2 - (n\pi)^2\right)\cos(n\pi) + 2n\pi\cancel{\sin(n\pi)} \right\}$$

$$b_n = \frac{2}{(n\pi)^3}\left\{ -2 + \left(2 - (n\pi)^2\right)(-1)^n \right\}$$

$$\therefore \quad b_n = \frac{-2}{(n\pi)}(-1)^n + \frac{4}{(n\pi)^3}\left((-1)^n - 1\right)$$

$$\therefore \quad f(x) \sim \sum_{n=1}^{\infty} \left\{ \frac{-2(-1)^n}{n\pi} + \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right] \right\} \sin(n\pi x)$$

(c) Same as usual...

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{where } L = 1$$

$$a_n = \frac{2}{1} \cdot \int_0^1 x^2 \cos(n\pi x)\, dx = \frac{4}{(n\pi)^2}(-1)^n \quad \text{if } n \neq 0.$$

$$\text{and} \quad a_0 = 2 \cdot \int_0^1 x^2 \cdot dx = \frac{2}{3}.$$

$$\therefore \quad f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}(-1)^n \cos(n\pi x)$$

(d) We see sine series have coefs that decay like $\frac{1}{n}$ whereas cosine series has coefs that decay like $\frac{1}{n^2}$ (faster). This is base on intuition of sine series for an $f(x)$ that is discontinuous at $x = -1, 1, 3, 5$, etc.

#4.

(a) Let $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$.

Multiply by e^{imx} and $\int_{-\pi}^{\pi} \cdot dx$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) e^{imx} \cdot dx = \sum_{n=-\infty}^{\infty} C_n \underbrace{\int_{-\pi}^{\pi} e^{i(n+m)x} dx}$$

$= 0$ if $n \neq -m$

and $= 2\pi$ if $n = -m$

Thus $C_{-m} = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x) e^{imx} dx$

or, letting $n = -m$

$$\Rightarrow \quad C_n = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x) e^{-inx} \cdot dx$$

Can verify same holds for $n = 0$.

(b) Mistake happens due to the fact

$$\tilde{e^{x}} = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

so LHS is the extension of e^x. We cannot differentiate LHS and expect equality as the extension is not even continuous at $x = n\pi$, n odd.

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

#4.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$= \underset{c_0}{\frac{a_0}{2}} + \frac{1}{2}(\underbrace{a_n - ib_n}_{c_n,\, n>0})e^{inx} + \frac{1}{2}(\underbrace{a_n + ib_n}_{c_n,\, n<0})e^{-inx}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$