1. $$\begin{cases} u_t = u_{xx} & x \in [0, L] \\ u_x(0,t) = 0 = u_x(L,t) \end{cases}$$

Let $u = X(x)\,G(t) \Rightarrow \frac{G'}{G} = \frac{X''}{X} = -\lambda^2 \le 0$

Chosen to avoid trivial solutions

$$\begin{cases} G' = -\lambda^2 G \\ X'' = -\lambda^2 X \end{cases}$$

If $\lambda = 0 \Rightarrow$ $G = A$ (1)

$X = Bx + C$ (2)

The boundary conditions $\Rightarrow u_x(0,t) = 0 \Rightarrow X'(0) = 0$

In (2), need $B = 0$ and

$\therefore$ only solution is that G, X are constant.

If $\lambda \neq 0$,

$G = \text{const} \times \exp(-\lambda^2 t)$

$X = A\cos \lambda x + B \sin \lambda x$

BC.s $\Rightarrow u_x(0,t) = 0 \Rightarrow X'(0) = 0 \Rightarrow B = 0.$

$u_x(L,t) = 0 \Rightarrow X'(L) = 0 \Rightarrow A\sin(\lambda L) = 0$

$\therefore \lambda_n = \frac{n\pi}{L}, \; n \in \mathbb{Z}.$

$\therefore$ Separable solutions :

$u_n(x,t) = A_n \exp(-\lambda_n^2 t)\cos(\lambda_n x)$

$\lambda_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}.$

Note that negative n are same as positive n (since $\cos(-z) = \cos z$) and $n=0$ case yields constant solution as expected.

Thus general solution:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right) \cos\left(\frac{n\pi x}{L}\right) \qquad (*)$$

(b) We seek to choose A_n so that

$$f(x) = u(x,0)$$

where $f(x)$ is an even function about $x=0$.

$$\therefore f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

This is just FS of the $[-L,L]$ even extension of $f(x)$ on $[0,L]$. Thus

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

The solution is $(*)$ with A_n given above.

Q2 First, need to solve the steady-state or time independent problem: let $u(x,t) = U(x)$

where
$$\begin{cases} U'' = 0 \\ U(0) = T_0 \\ U(L) = T_1 \end{cases}$$

so $U = Ax + B \Rightarrow U = \left(\frac{T_1 - T_0}{L}\right) \cdot x + T_0$.

(a,b) $T_0 = 0, T_1 = 1, \Rightarrow U = \left(\frac{1}{L}\right) x$.

Now that we have solved for $U(x)$, set,

$$u(x,t) = U(x) + \hat{u}(x,t).$$

Thus $\hat{u}(x,t)$ satisfies:
$$\begin{cases} \hat{u}_t = \hat{u}_{xx} \\ \hat{u}(0,t) = 0 = \hat{u}(L,t) \\ \hat{u}(x,0) = f(x) - U(x) \end{cases}$$

In (a) $f(x) = 0 \Rightarrow \hat{u}(x,0) = 0 - \frac{x}{L}$

In (b) $f(x) = \sin\left(\frac{\pi x}{L}\right) \Rightarrow \hat{u}(x,0) = \sin\left(\frac{\pi x}{L}\right) - \frac{x}{L}$.

The solution is given by Thm. 14.1

$$\hat{u}(x,t) = \sum_{n=1}^{\infty} B_n \sin(\lambda_n x) \exp(-\lambda_n^2 t) \qquad \lambda_n = \frac{n\pi}{L}$$

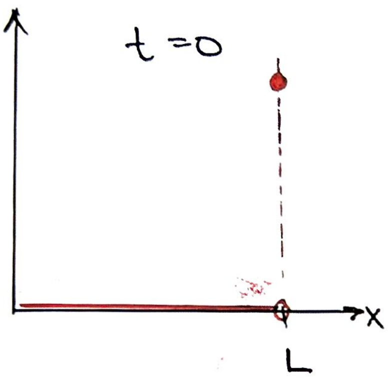
$$B_n = \frac{2}{L} \cdot \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

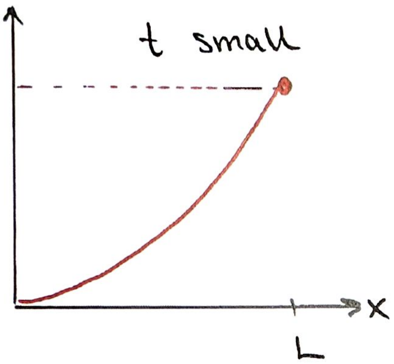
So in (a) $B_n = \frac{2}{L} \int_0^L \left(-\frac{x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$

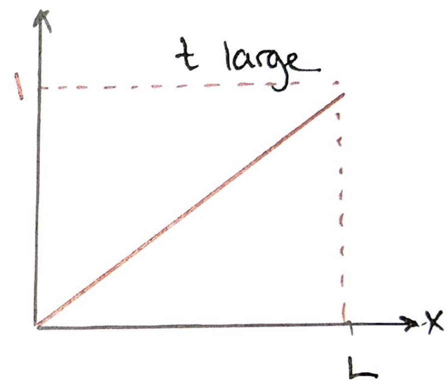
$$= \frac{2}{n\pi} (-1)^n .$$

$\therefore \; u(x,t) = U(x) + \hat{u}(x,t)$

$$= \frac{x}{L} + \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^n \exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right) \cdot \sin\left(\frac{n\pi x}{L}\right)$$







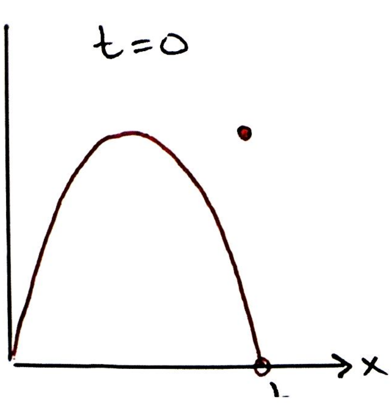
(b) This part similar. Need to integrate

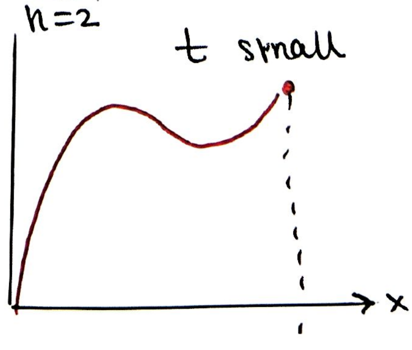
$$B_n = \frac{2}{L} \int_0^L \left\{ \sin\left(\frac{\pi x}{L}\right) - \frac{x}{L} \right\} \sin\left(\frac{n\pi x}{L}\right) dx$$

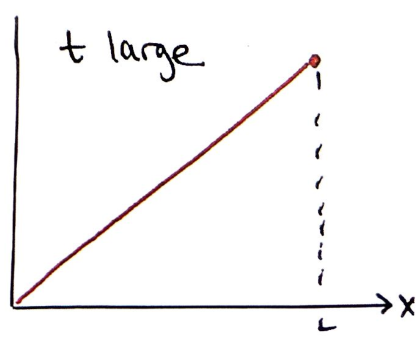
$$= \begin{cases} 1 - \frac{2}{\pi} & \text{if } n = 1 \\ \frac{2}{n\pi}(-1)^n & n = 2, 3, 4, \ldots \end{cases}$$

Thus, $u(x,t) = \frac{x}{L} + \left(1 - \frac{2}{\pi}\right) \exp\left(-\left(\frac{\pi}{L}\right)^2 t\right) \sin\left(\frac{\pi x}{L}\right)$

$$+ \sum_{n=2}^{\infty} \frac{2}{n\pi} (-1)^n \exp\left(-\left(\frac{n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right)$$







Q3.

What happens if we don't assume the constant is negative. Consider Dirichlet problem with

$$u(0,t) = 0 = u(L,t).$$

Usual sep. of vars $\Rightarrow u = X(x)G(t)$

$$\Rightarrow \frac{G'}{G} = \frac{X''}{X} = \lambda^2 > 0$$

We wonder does lead to a contradiction?

Then $G(t) = \text{const} \cdot \exp(\lambda^2 t)$ ← exp. growth

$X(x) = A\exp(\lambda x) + B\exp(-\lambda x)$

$$\left.\begin{aligned} X(0) &= 0 \Rightarrow A + B = 0. \\ X(L) &= 0 \Rightarrow A\exp(\lambda L) + B\exp(-\lambda L) = 0 \end{aligned}\right\}$$

Solving $\Rightarrow A = B = 0 \Rightarrow$ trivial solution.

* What happens if $\lambda = 0$?

Then $X(x) = Ax + B$ and again $A = 0 = B$ by the boundary conditions

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* What happens for the Neumann condition,

$$u_x(0,t) = 0 = u_x(L,t) \quad ?$$

We already explained the case of $\lambda = 0$ in Q1. If $\lambda \neq 0$, then again $X(x) = A\exp(\lambda x) + B\exp(-\lambda x)$ and the only solution with $X'(0) = 0 = X'(L)$ is $A = 0 = B$.

#4. We seek to solve $T_{rr} + \frac{1}{r} T_r + \frac{1}{r^2} T_{\theta\theta} = 0$.

Let $T(r,\theta) = F(r) \cdot G(\theta)$ gives,

$$\frac{r^2 F'' + rF'}{F} = -\frac{G''}{G} = \sigma$$

So need to solve $\begin{cases} G'' = -\sigma G \\ r^2 F'' + rF' - \sigma F = 0. \end{cases}$

However, need $G(\theta)$ periodic $\Rightarrow G(\theta + 2\pi) = G(\theta)$. Assume that $\sigma > 0$. Then

$$G(\theta) = C_1 \cos(\sqrt{\sigma}\,\theta) + C_2 \sin(\sqrt{\sigma}\,\theta).$$

For periodicity, need $\sqrt{\sigma} = n \in \mathbb{Z} \Rightarrow \sigma = n^2$.

What happens if $\sigma = 0$ or $\sigma < 0$? If $\sigma = 0$, then $G = C_1\theta + C_2$ so need $C_1 = 0$. Similarly if $\sigma < 0$, $G = C_1 \exp(\sigma\theta) + C_2 \exp(-\sigma\theta)$ and periodicity requires $C_1 = C_2 = 0$.

$\therefore$ Case 1 : $\sigma = n^2$ and $G = C_1 \cos(n\theta) + C_2 \sin(n\theta)$

where $n = 1, 2, 3, 4, \ldots$

Case 2 : $\sigma = 0$ and $G = C_3$.

Now consider $F(r)$ in both cases.

Case 2: $\sigma = 0$ and $r^2F'' + rF' = 0$.

$$\Rightarrow rF'' + F' = 0$$

$$\Rightarrow \frac{d}{dr}(rF') = 0$$

$$\therefore F = D_1 + D_2 \log r$$

Thus need $D_2 \equiv 0$ so that $F(r)$ bounded as $r \to 0$. We conclude that both $G(\theta)$ and $F(r)$ are constant.

Case 1: $\sigma = n^2 \Rightarrow r^2 F'' + rF' - n^2 F = 0$.

Let $F = r^\alpha \Rightarrow \alpha(\alpha-1) r^\alpha + \alpha . r^\alpha - n^2 r^\alpha = 0$.

$$\therefore \alpha(\alpha - 1) + \alpha - n^2 = 0$$

$$\alpha = \pm n$$

$$\therefore$$

$$\therefore F(r) = D_1 r^n + D_2 . r^{-n}$$

$$G(\theta) = C_1 \cos n\theta + C_2 \sin n\theta$$

So in conclusion seperable solutions follow

$$T_n(r,\theta) = \left(C_1 \cos n\theta + C_2 \sin n\theta\right)\left(D_1 r^n + D_2 r^{-n}\right)$$

Finally, we need $D_2 \equiv 0$ in order for solutions defined at $r = 0$, $n > 0$

$$\Rightarrow T_n(r,\theta) = \left(A \cos n\theta + B \sin n\theta\right) r^n \qquad n = 0, 1, 2, 3, \ldots$$

b) Solve $\nabla^2 T = 0$ with $T(a,\theta) = \begin{cases} 1 & 0 \le \theta < \pi \\ 0 & \pi \le \theta < 2\pi \end{cases}$

use $T(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos n\theta + B_n \sin n\theta\right) r^n$

note your Fourier coefs are
"a_n" $= A_n r^n$
"b_n" $= B_n r^n$

$$A_n \cdot a^n = \frac{1}{\pi} \cdot \int_0^{2\pi} T(a,\theta) \cos n\theta \cdot d\theta \qquad n \geq 0.$$

$$a^n A_n = \frac{1}{\pi} \cdot \int_0^{\pi} \cos n\theta \cdot d\theta = 0 \qquad \forall\ n > 0.$$

check $n=0$: $\quad A_0 = \frac{1}{\pi} \cdot \int_0^{\pi} d\theta = 1$

Also $\quad B_n \cdot a^n = \frac{1}{\pi} \cdot \int_0^{\pi} \sin n\theta \cdot d\theta$

$$= \frac{1}{\pi} \left[-\frac{\cos n\theta}{n} \right]_0^{\pi}$$

$$= \frac{1}{n\pi} \cdot \left[1 - (-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

$$\therefore\ T(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n} \cdot \frac{1}{n\pi} \left[1 - (-1)^n\right] \cdot \sin n\theta$$

let $n = 2m+1$

$$= \frac{1}{2} + \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^{2m+1} \frac{2}{(2m+1)\pi} \sin\left[(2m+1)\theta\right]$$