Problem set 9: The wave equation

1. (a) The procedure is identical to the one in the notes for the Dirichlet problem, up to the imposition of the Neumann boundary conditions. By separation of variables, $u = X(x)G(t)$, one obtains that

$$\frac{G''}{c^2 G} = \frac{X''}{X} = -\lambda^2 \leq 0.$$

The inclusion of $\lambda = 0$ will be necessary in a moment. Thus it follows that

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

By the Neumann conditions, we need firstly that $X'(0) = 0$, and this implies that $B = 0$. Secondly, $X'(L) = 0$ and this implies that

$$\lambda A \sin(\lambda L) = 0 \implies \lambda L = n\pi \implies \lambda_n = \frac{n\pi}{L},$$

for $n \in \mathbb{Z}$. The solution for the $G(t)$ follows directly from solving $G'' + (c\lambda)^2 G = 0$, and we obtain the requisite family of solutions

$$u_n(x,t) = \cos\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right).$$

There is a small issue that we do not require the negative integers $\ldots, -3, -2, -1$. If $n = -m < 0$, then we see that by the even or odd properties of cosine and sine, respectively, a negative integer merely produces the same form as the positive integer but with the coefficient in front of the sine negated. The arbitrary nature of A_n and B_n handles this.

The case of $\lambda = 0$ must be treated separately. Here, we see

$$X'' = 0 \Rightarrow X = K_0 + K_1 t.$$

However $X'(0) = X'(L) = 0$ implies $K_1 = 0$. Similarly,

$$G'' = 0 \Rightarrow G = U_0 + U_1 t.$$

Thus we must also allow for the mode

$$u_0(x,t) = K_0(U_0 + U_1 t) = \frac{A_0}{2} + \frac{B_0 t}{2}.$$

(The factor of half chosen to simplify the Fourier expansions.)

(b) Assume for simplicity $L = \pi$. The function is $\cos(x)\cos(ct)$, thus the temporal period is $T = 2\pi/c$. The graphs at the three times $t = \{0, T/2, T\}$ correspond to $\{\cos(x), -\cos(x), \cos(x)\}$.

(c) Assume for simplicity $L = \pi$. The function is $\cos(2x)\cos(2ct)$, thus the temporal period is $T = 2\pi/(2c)$. The graphs at the three times $t = \{0, T/2, T\}$ correspond to $\{\cos(2x), -\cos(2x), \cos(2x)\}$.

(d) The general solution is thus written as

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right).$$

At $t = 0$, we have for the displacement,

$$u_0(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

so the A_n coefficients are given by

$$A_n = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{n\pi x}{L}\right), \quad \text{for } n = 0, 1, 2, \dots$$

Similarly, for the velocity,

$$v_0(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \cos\left(\frac{n\pi x}{L}\right).$$

It is easier if we write this as

$$v_0(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right).$$

where we have set $b_0 = B_0$ and $b_n = B_n(n\pi c/L)$ for $n \geq 1$. Thus we conclude that

$$b_n = \frac{2}{L}\int_0^L v_0(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Or in terms of B_n, we have

$$B_0 = \frac{2}{L}\int_0^L v_0(x)\,\mathrm{d}x,$$
$$B_n = \frac{2}{L}\frac{L}{n\pi c}\int_0^L v_0(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \quad n \geq 1$$

2. Same procedure as usual, resulting in

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2hL^2}{(n\pi)^2(L-p)} \sin\left(\frac{n\pi p}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).$$

3. This is exactly from the course notes (Theorem 18.3 in 18-19').

4. See attached.

#4

The Q is difficult because of numbers / variables.

Note:

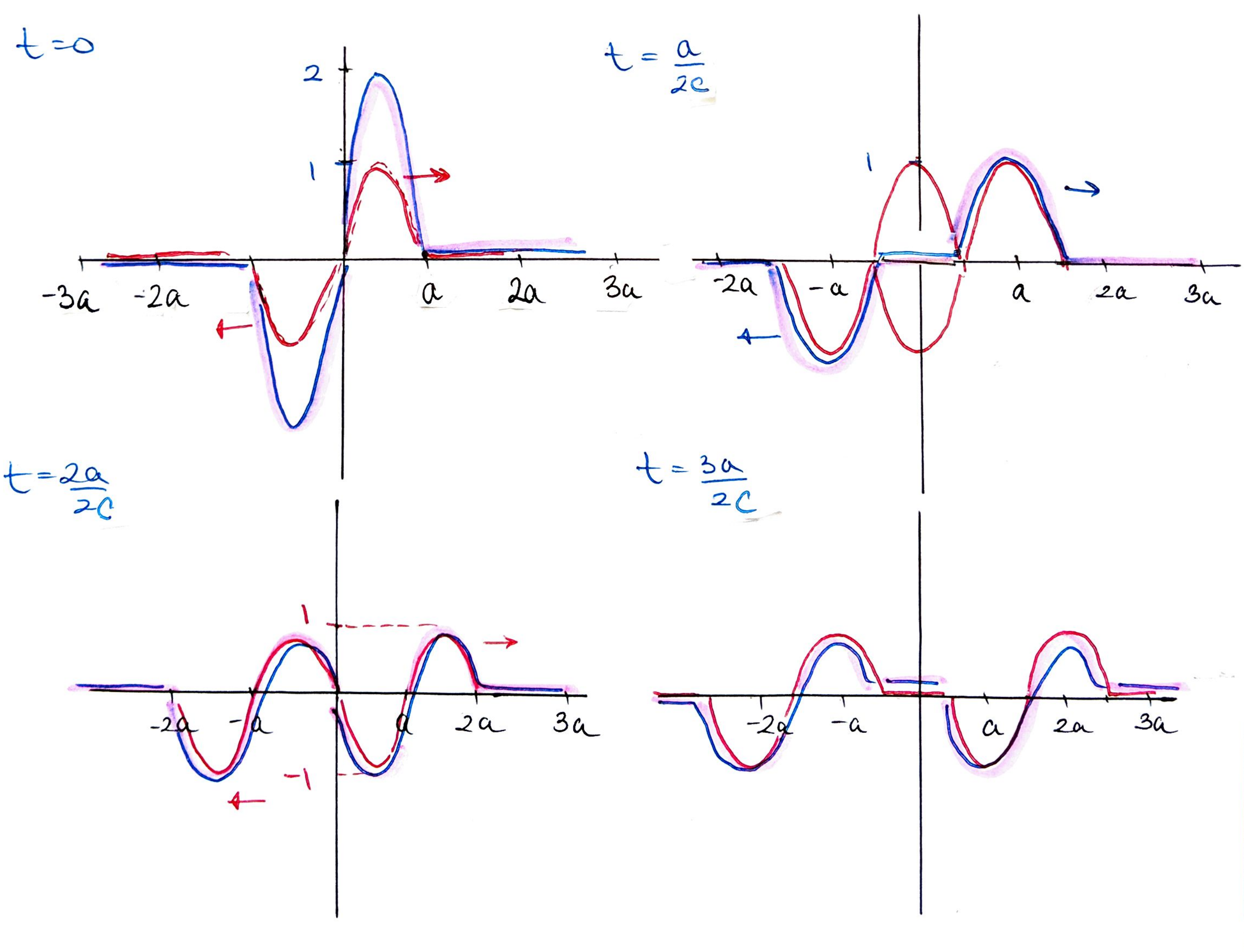
* For $t > 0$, initial wave splits into left + right travelling waves of half the initial height

* The sine wavelength is $\frac{2\pi}{\pi/a} = 2a = \lambda$

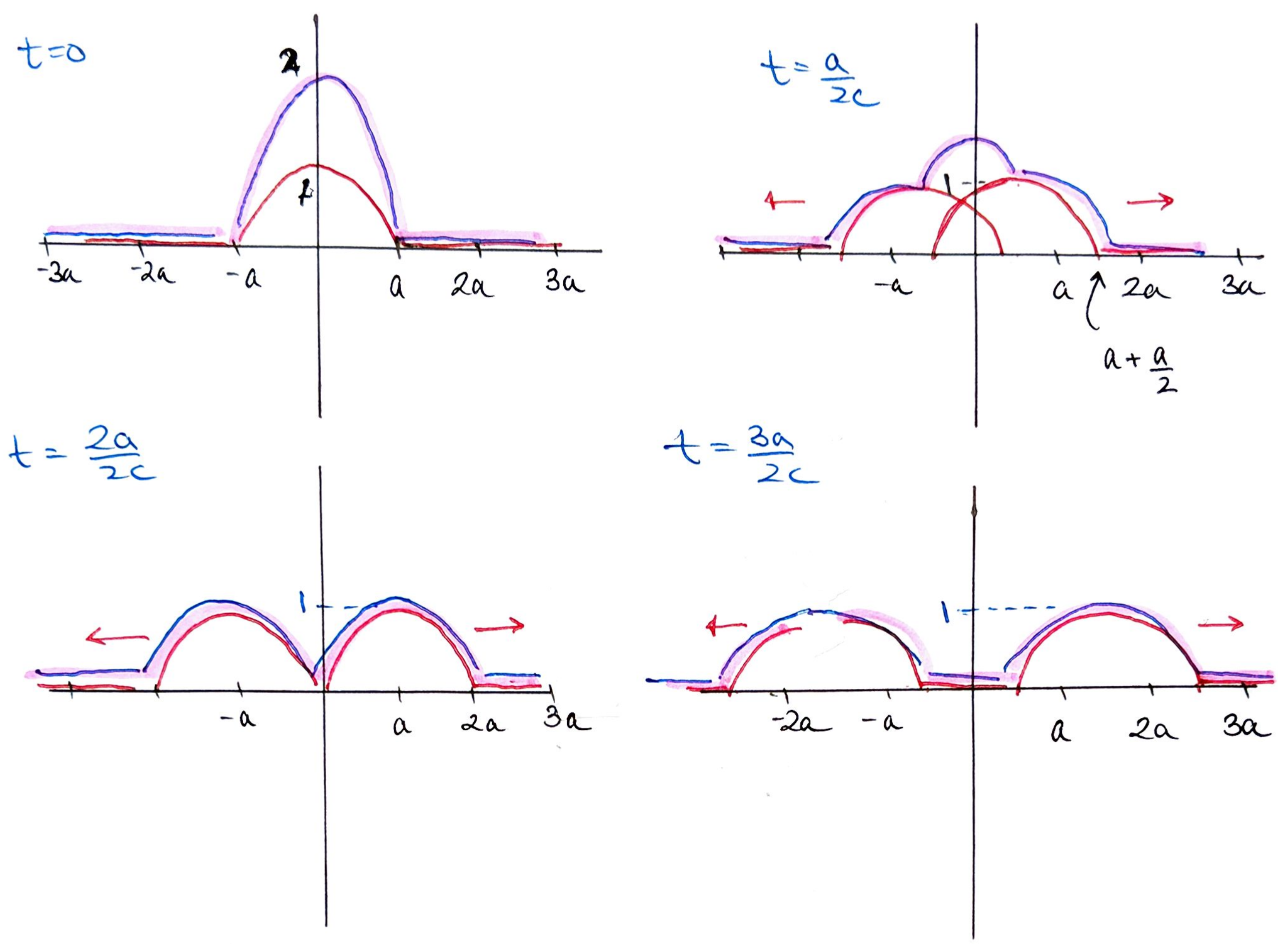
* The times requested are $\frac{a}{2c}, \frac{2a}{2c}, \frac{3a}{2c}$, etc.

* The speed is c. So in time $\frac{a}{2c}$, the wave moves

$$\frac{a}{2} = \frac{\lambda}{4} = 1/4 \text{ of wavelength}$$



(b) Same deal but with $u(x,0) = \begin{cases} \cos\left(\frac{\pi x}{2a}\right) & |x| \le a \\ 0 & |x| > a \end{cases}$



SUPPLEMENTARY

5. The case of $C = \lambda^2 > 0$ results in $X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$. Then the boundary conditions $X(0) = 0$ implies $A = 0$ and the boundary condition $X(L) = 0$ results in $B = 0$. Similarly, the case that $X'' = 0$ results in the trivial solution. The case of Neumann conditions allows for the solution used in Q1.

6. Setting the Dirichlet conditions at $u = 0$ corresponds to fixing the height of the string at the respective point. The most intuitive interpretation of the Neumann condition, $u_x(0, t) = F_0$, for instance, is that this clamps the string so that it takes a particular angle at the respective boundary.

   A more comprehensive answer seeks to understand the connection between Neumann condition and force. Note that in our derivation of the wave equation, we showed that the vertical force at a point is given by

   $$T \sin\theta \approx T\theta,$$

   where T is the tension in the string, and where the approximation occurs since θ is small, and $\sin\theta$ behaves as θ for small values. However, also as a consequence of θ being small, $\theta \approx \tan\theta$ (again, consider the Taylor series of $\tan\theta$ as $\theta \to 0$). Finally, $\tan\theta \approx \frac{\partial u}{\partial x}$ by elementary geometry and definition of the tangent as the 'opposite over adjacent'. Hence,

   $$\text{vertical force} \approx T\frac{\partial u}{\partial x}.$$

   Fixing the value of u_x then corresponds to imposing a vertical force.