Vector calculus & partial differential equations

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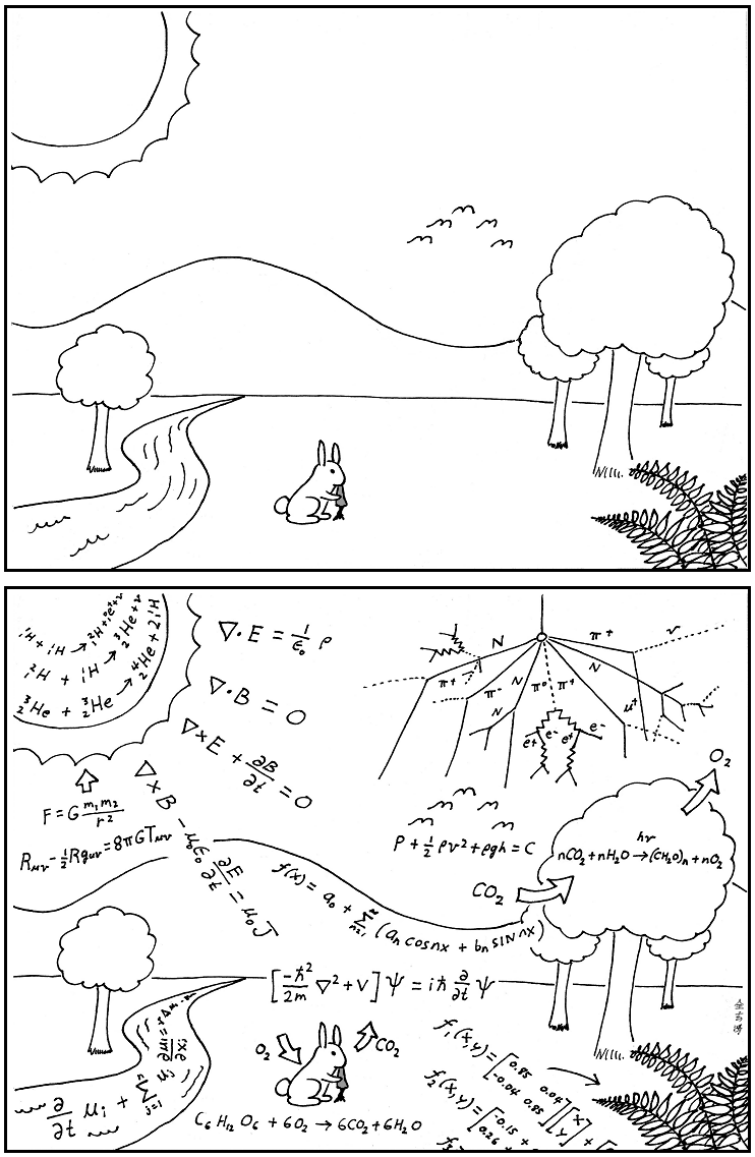

A lecture workbook for 2019–20'

P.H. Trinh, draft February 1, 2020

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Contents

Chapter 1

The first lecture

This lecture will introduce the key reasons why we should be studying vector calculus and partial differential equations.

Chapter 2

A review of multivariable calculus

In this chapter, we review some of the basic ideas of integration in multiple dimension, notably the concept of changing integration in cartesian coordinates to another (curvilinear) coordinate system. We will then review the notion of the gradient operator, and the related ideas of conservative fields and potentials.

Example 2.1 (Examples of scalar and vector fields)**.** Vector fields include velocity $\boldsymbol{v}(\boldsymbol{x})$, electric field $\boldsymbol{E}(\boldsymbol{x})$, magnetic field $\boldsymbol{B}(\boldsymbol{x})$, gravitational field $\boldsymbol{G}(\boldsymbol{x})$. Scalar fields include speed $|\boldsymbol{v}(\boldsymbol{x})|$, kinetic energy $\frac{1}{2}m|\boldsymbol{v}(\boldsymbol{x})|^2$, electric potential $\phi_E(\boldsymbol{x})$.

Definition 2.2 (Scalar and vector fields)**.** A function $f : \mathbb{R}^3 \to \mathbb{R}$ is called a *scalar field*. A function $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is called a *vector field*. When we need to split $\boldsymbol{F}$ into components, we write $\boldsymbol{F} = (F_1, F_2, F_3)$, i.e. $(\boldsymbol{F})_i = F_i$ for $i = 1, 2, 3$.

We will mostly work in 2- or 3-dimensions, but most of these concepts can be extended to higher dimensions. Thus we write $\boldsymbol{x} = (x, y)$ in 2-d and $\boldsymbol{x} = (x, y, z)$ in 3-d. We let $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ be the unit vectors in the x, y, and z directions respectively.

2.1 Double and triple integrals

Double integrals

Notation 2.3 (Double integrals)**.** If $\Omega \subset \mathbb{R}^2$ and f is a scalar function on Ω, then

$$\int_\Omega f \,\mathrm{d}A = \iint_\Omega f(x, y) \,\mathrm{d}x \,\mathrm{d}y,$$

with the A in $\mathrm{d}A$ standing for "area".

In this course, we will generally use the convention of explicitly writing multiple number of integral signs unless it is terribly inconvenient to do so.

Example 2.4 (Physical example of a double integral)**.** If $\Omega \subset \mathbb{R}^2$, the volume under a 2-d surface $f(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$ is given by $\int_\Omega f(\boldsymbol{x}) \,\mathrm{d}A$. This interpretation of volume assumes $f \geq 0$. Compare with the interpretation of a one-dimensional integral as two-dimensional area.

Definition 2.5 (Definition of double integral as a Riemann sum)**.** Given $\Omega \subset \mathbb{R}^2$, subdivide Ω into N subregions $A_1, \ldots, A_N$ with areas $\delta A_1, \ldots, \delta A_N$. In each of the subregions choose an arbitrary point $\boldsymbol{x}_i$. Then, for any scalar function f on Ω,

$$\iint_\Omega f(\boldsymbol{x}) \,\mathrm{d}A \equiv \lim_{N\to\infty} \sum_{i=1}^{N} f(\boldsymbol{x}_i)\, \delta A_i,$$

provided that this limit exists and is independent both of the choice of $\boldsymbol{x}_i$ and of the particular subdivision.

(Double) integration follows the usual linearity and additivity properties. See first-year lecture notes to review the basic properties. In our case, one of the more important properties allows you to convert between coordinate systems using the Jacobian.

Theorem 2.6 (Change of variables)**.** If the transformation $x = x(u, v)$, $y = y(u, v)$ maps S in the (u, v)-plane to R in the (x, y)-plane, then

$$\iint_R f(x, y) \,\mathrm{d}x \,\mathrm{d}y = \iint_S f\big(x(u, v), y(u, v)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u \,\mathrm{d}v,$$

where the *Jacobian* $\partial(x, y)/\partial(u, v)$ is defined by

$$\frac{\partial(x, y)}{\partial(u, v)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Example 2.7 (Jacobian in polar coordinates)**.** If $x = r\cos\theta$ and $y = r\sin\theta$, then $\partial(x, y)/\partial(r, \theta) = r$ so $\mathrm{d}x \,\mathrm{d}y = r \,\mathrm{d}r \,\mathrm{d}\theta$.

Definition 2.8 (Double integrals of vector fields)**.** If $\Omega \subset \mathbb{R}^2$ and $\boldsymbol{F}$ is a vector function on Ω, then

$$\int_\Omega \boldsymbol{F} \,\mathrm{d}A = \begin{pmatrix} \int_\Omega F_1 \,\mathrm{d}A \\ \int_\Omega F_2 \,\mathrm{d}A \\ \int_\Omega F_3 \,\mathrm{d}A \end{pmatrix} \quad \text{i.e.} \quad \left(\int_\Omega \boldsymbol{F} \,\mathrm{d}A\right)_i = \int_\Omega F_i \,\mathrm{d}A. \tag{2.1}$$

Triple integrals

Notation 2.9 (Triple integrals)**.** If $\Omega \subset \mathbb{R}^3$,

$$\int_\Omega f(\boldsymbol{x}) \,\mathrm{d}V = \iiint_\Omega f(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z,$$

with the V in $\mathrm{d}V$ standing for "volume".

We consider triple integrals via the usual definition in terms of a Riemann sum, where the properties of linearity, additivity, evaluation by successive integration, and their definition for vector fields are all analogous to the double-integral case.

Example 2.10 (Physical example of a triple integral)**.** If the density of a body, Ω, is given by $\rho = \rho(\boldsymbol{x})$, then the mass of the body is given by $\int_\Omega \rho(\boldsymbol{x}) \,\mathrm{d}V$.

Definition 2.11 (Spherical polar coordinates)**.**

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

for $0 < r < \infty$, $0 \le \theta < 2\pi$, and $0 \le \phi < \pi$.

In order to convert between coordinate systems, we recall that the Jacobian is defined by the determinant,

$$J = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| := \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}.$$

Theorem 2.12 (Integration in spherical coordinates)**.** The Jacobian for spherical coordinates is given by

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2 \sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Thus for a given $\Omega \subset \mathbb{R}^3$,

$$\iiint_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\Omega_{r\theta\phi}} g(r,\theta,\phi)\,r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi,$$

where the integrand has been changed using,

$$g(r,\theta,\phi) = f\big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big)$$

for integration in the new domain $\Omega_{r\theta\phi}$.

2.2 Directional derivatives, gradients, and potentials

Directional derivatives and gradients

Definition 2.13 (Gradient)**.** Let $\Omega \subset \mathbb{R}^3$ and let $f : \Omega \to \mathbb{R}$ be a differentiable scalar field. Then the *gradient* of f on Ω is defined by

$$\nabla f \;:=\; \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}. \tag{2.2}$$

Note that f is a scalar field while ∇f is a vector field. The gradient is also sometimes written as $\mathrm{grad} f$.

Lemma 2.14 (Basic properties of the gradient)**.** If f and g are differentiable scalar fields and λ and μ are constants, then

$$\nabla(\lambda f + \mu g) = \lambda\nabla f + \mu\nabla g, \tag{2.3}$$

and

$$\nabla(fg) = f\nabla g + g\nabla f. \tag{2.4}$$

Idea for the proof of Lemma 2.14. For (2.3) split into components and use linearity of differentiation. For (2.4) split into components and use the product rule.

The gradient quantities the rate-of-change of f in each of the Cartesian directions. In order to obtain the rate-of-change in a *particular* direction, we define the *directional derivative*.

Definition 2.15 (Directional derivative)**.** Let $\Omega \subset \mathbb{R}^3$ and $f : \Omega \to \mathbb{R}$ be a differentiable scalar field. For a vector, $\boldsymbol{a}$ in $\mathbb{R}^3$, define the unit vector $\widehat{\boldsymbol{a}} := \boldsymbol{a}/|\boldsymbol{a}|$. Then the *directional derivative* of f in the direction $\widehat{\boldsymbol{a}}$ at $\boldsymbol{x}_0 \in \Omega$ is defined by

$$D_{\boldsymbol{a}} f(\boldsymbol{x}_0) \;:=\; \lim_{h\to 0} \frac{f(\boldsymbol{x}_0 + h\widehat{\boldsymbol{a}}) - f(\boldsymbol{x}_0)}{h}. \tag{2.5}$$

Thus by definition, $D_{\boldsymbol{a}} f(\boldsymbol{x}_0)$ is the rate of change of f in the direction of $\widehat{\boldsymbol{a}}$ at the point $\boldsymbol{x}_0$.

The link between the directional derivative and the gradient comes from the next important lemma.

Lemma 2.16 (Link between directional derivative and gradient)**.**

$$D_{\boldsymbol{a}} f(\boldsymbol{x}_0) = \nabla f(\boldsymbol{x}_0) \cdot \widehat{\boldsymbol{a}} = |\nabla f(\boldsymbol{x}_0)| \cos\theta, \tag{2.6}$$

where θ is the angle between $\nabla f(\boldsymbol{x}_0)$ and $\widehat{\boldsymbol{a}}$.

Idea for the proof of Lemma 2.16 Use the fact that

$$D_{\boldsymbol{a}} f(\boldsymbol{x}_0) = \left(\frac{\mathrm{d}}{\mathrm{d}s} f\big(\boldsymbol{x}_0 + s\widehat{\boldsymbol{a}}\big) \right)\Bigg|_{s=0}$$

and then use the chain rule.

Note that if $\nabla f(\boldsymbol{x}_0) = 0$ then the directional derivatve $D_{\boldsymbol{a}} f(\boldsymbol{x}_0) = 0$ for every direction $\boldsymbol{a}$. Then we say that f has a *stationary point* (the analogue of a point where $f'(x_0) = 0$ for an $f : \mathbb{R} \to \mathbb{R}$).

Corollary 2.17 (Gradient as direction of steepest descent)**.** If $\nabla f(\boldsymbol{x}_0) \neq \boldsymbol{0}$, then $\boldsymbol{a} = \nabla f$ points in the direction of steepest descent or ascent. This can be written as

$$\max_{\boldsymbol{a} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}} \big| D_{\boldsymbol{a}} f(\boldsymbol{x}_0) \big| = |\nabla f(\boldsymbol{x}_0)|. \tag{2.7}$$

If $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$ then the maximum is zero and is attained in every direction.

Proof of Corollary 2.17. From (2.6) we see that the modulus of the left-hand side is maximised when θ is an integer multiple of π, i.e. when $\widehat{\boldsymbol{a}}$ and ∇f point in the same direction or 180° from one another.

Corollary 2.18 (Gradient as perpendicular to level sets)**.** The gradient, ∇f, is a vector perpendicular to the level sets, $f(\boldsymbol{x}) = c$, where c is constant.

Proof of Corollary 2.18. Consider some point, $\boldsymbol{x}_0$, that lies along a level set. Thus $f(\boldsymbol{x}_0) = c$. Take a vector, $\boldsymbol{a}$, tangent to the level set. Then since f does not change, $D_{\boldsymbol{a}} f(\boldsymbol{x}_0) = 0$. Then (2.6) implies that $\widehat{\boldsymbol{a}} \cdot \nabla f(\boldsymbol{x}_0) = 0$, i.e. $\nabla f(\boldsymbol{x}_0)$ is perpendicular to $\widehat{\boldsymbol{a}}$. Since this holds for any tangent vector at $\boldsymbol{x}_0$, we must have that $\nabla f(\boldsymbol{x}_0)$ is perpendicular to the surface.

Example 2.19 (Example of Corollary 2.18)**.** If $f(\boldsymbol{x}) = x^2 + y^2 + z^2$ then $f(\boldsymbol{x}) = a^2$ (with a constant) defines a sphere of radius a. We have $\nabla f = (2x, 2y, 2z) = 2\boldsymbol{x}$, which is perpendicular (i.e. normal) to the surface of the sphere.

Conservative fields and potentials

Many physical problems involve the study of *conservative* forces, represented by a vector field $\boldsymbol{F}(\boldsymbol{x})$. There are a variety of ways to define the conservative property, but physically, $\boldsymbol{F}$ is conservative if the work done in moving a particle from position A to position B is independent from the path taken. It is important to remember that different settings and references may present a different definition of being conservative.

However, we shall define the conservative property for a vector field in a different way: via the existence of a scalar potential, ϕ. In the next chapter and in Theorem 4.3, we will present the various equivalences of being "conservative", which serves to unify different definitions.

Definition 2.20 (Conservative fields and potentials)**.** The vector field $\boldsymbol{F}$ is called *conservative* if there exists a scalar field, ϕ, defined on a simply-connected region such that $\boldsymbol{F} = \nabla\phi$.

Note that if ϕ is a scalar potential for $\boldsymbol{F}$, then $\phi + c$ is also a scalar potential for $\boldsymbol{F}$ for every constant $c \in \mathbb{R}$. Therefore a potential is not unique.

Example 2.21 (Example of a potential)**.** The gravitational field

$$\boldsymbol{G}(\boldsymbol{x}) := -\frac{GM}{|\boldsymbol{x}-\boldsymbol{x}_0|^3}(\boldsymbol{x}-\boldsymbol{x}_0)$$

is conservative since

$$\nabla\left(\frac{1}{|\boldsymbol{x}|}\right) = -\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}$$

and so

$$\boldsymbol{G}(\boldsymbol{x}) = \nabla\left(GM\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_0|}\right).$$

Chapter 3

The line integral

3.1 Vector equation of curves

Definition 3.1 (Curve)**.** By a **curve** in $\mathbb{R}^3$, typically denoted C, we shall mean a piecewise smooth function, $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$. In component form, we write $\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)$ and call t the **parameter**. Note that order on the interval also gives the curve an **orientation**.

Definition 3.2 (Simple and closed)**.** (i) A curve is *simple* if it does not intersect itself; i.e. if for a given $t_1, t_2 \in [a, b]$ with $t_1 \neq t_2$ then $\boldsymbol{r}(t_1) \neq \boldsymbol{r}(t_2)$. (ii) A curve is said to be *closed* if $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.

Lemma 3.3 (Tangent vector)**.** $(\mathrm{d}\boldsymbol{r}/\mathrm{d}t)(t_0)$ is a vector tangent to the curve at t_0.

Proof. Proof by picture.

Lemma 3.4 (Arc length)**.** The arc length of a simple curve is defined to be

$$s(t) \equiv \int_a^t \left| \frac{\mathrm{d}\boldsymbol{r}(t')}{\mathrm{d}t'} \right| \mathrm{d}t'. \tag{3.1}$$

Proof. An infinitessimal arclength element is given by $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2$. Hence we divide by $\mathrm{d}t$ and it follows that

$$\mathrm{d}s = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,\mathrm{d}t = |\boldsymbol{r}'(t)|\,\mathrm{d}t. \tag{3.2}$$

The result follows by summing the above from a to t. The above can be justified more formally using the chain rule.

Example 3.5 (Parameterisation of a circle)**.** We can parameterize a circle of radius a using

$$\boldsymbol{r}(\theta) = a(\cos\theta, \sin\theta),$$

for $\theta \in [0, 2\pi]$. Here, $\mathrm{d}s/\mathrm{d}\theta = a$, so for a unit change in θ, the arclength increases by a units. How would you choose the parameterisation, say with t, so that $\mathrm{d}s/\mathrm{d}t = 1$?

Note that when we parameterize the curve $\boldsymbol{r}(t)$ using t, the curve may lengthen in some complicated fashion as t increases. Certain choices of how to measure the curve using t may be more convenient. For instance, t can be chosen to correspond to arclength as measured from the initial point. If (3.1) is differentiated, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|.$$

Thus if we set $t = s$ (i.e. by parametrising by arclength) we have

$$\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\right| = 1 \quad \text{for all } s. \tag{3.3}$$

Definition 3.6 (Reversed curve $-C$)**.** Given a curve $C \subset \mathbb{R}^3$ with parameterisation $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$, let $-C$ denote the curve C with the parameterisation $\widetilde{\boldsymbol{r}} : [-b, -a] \to \mathbb{R}^3$ defined by $\widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(-t)$

3.2 Line integrals of scalar fields

Definition 3.7 (Line integral of a scalar field)**.** Let C be a curve with parametrisation $\boldsymbol{r}(t)$ for $a \le t \le b$. The *line integral* of a scalar field f along C is defined by

$$\int_C f \,\mathrm{d}s := \int_a^b f(\boldsymbol{r}(t)) \left|\boldsymbol{r}'(t)\right| \mathrm{d}t. \tag{3.4}$$

This definition follows naturally from substitution of the Pythagoras expression for the arclength in (3.2) into the integrand $f\,\mathrm{d}s$. If the curve C is closed, then we often write the line integral as

$$\oint_C f \,\mathrm{d}s. \tag{3.5}$$

One can show that the value of the integral is independent of the parametrisation of the curve.

Example 3.8. Calculate the line integral $\int_C f\,\mathrm{d}s$ when $f(\boldsymbol{x}) = x + y + z$ and the curve C is the straight line joining the points $(1, 2, 3)$ and $(4, 5, 6)$.

Solution. We parameterize the straight line between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{r}(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$ for $t \in [0, 1]$. Therefore, for our C we have $\boldsymbol{r}(t) = (1, 2, 3) + t(3, 3, 3) = (1 + 3t, 2 + 3t, 3 + 3t)$. From (3.4), we have

$$\int_C f \,\mathrm{d}s := \int_0^1 f(\boldsymbol{r}(t)) \left|\boldsymbol{r}'(t)\right| \mathrm{d}t = \int_0^1 \Big((1 + 3t) + (2 + 3t) + (3 + 3t)\Big) \left|\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}\right| \mathrm{d}t.$$

Since $|(3, 3, 3)| = 3|(1, 1, 1)| = 3\sqrt{3}$, we have

$$\int_C f \,\mathrm{d}s = 3\sqrt{3} \int_0^1 (6 + 9t)\mathrm{d}t = 3\sqrt{3}\left(6 + \frac{9}{2}\right).$$

Lemma 3.9 (Basic properties of line integrals of scalar fields)**.**

1. *Linearity:* if λ and μ are constants and f and g are scalar fields, then

$$\int_C (\lambda f + \mu g) \,\mathrm{d}s = \lambda \int_C f \,\mathrm{d}s + \mu \int_C g \,\mathrm{d}s.$$

2. *Additivity w.r.t. curve* if $C = C_1 \cup C_2$ then

$$\int_C f \, \mathrm{d}s = \int_{C_1} f \, \mathrm{d}s + \int_{C_2} f \, \mathrm{d}s.$$

3. *Independence of direction:* with $-C$ defined by Definition 3.6,

$$\int_{-C} f \, \mathrm{d}s = \int_C f \, \mathrm{d}s. \tag{3.6}$$

The first two properties follow directly from the definition (3.4); you will prove the 3rd property on PS1 Q2.

3.3 Line integrals of vector fields

The extension of the scalar line integrals in the previous section to vector line integrals falls into two types. The first type is of the form

$$\int_C \boldsymbol{F} \, \mathrm{d}s. \tag{3.7}$$

Notice that both integrands are written in a shorthand, with the parameterisation variable, say t, not explicitly written. You will get in the habit of converting these integrals to a computational form.

Thus in the integral above, we take the value of the vector $\boldsymbol{F}$ at each point along a curve and multiply by an arclength element, then sum each product. The result is a vector. It is defined formally as follows.

Definition 3.10 (Vector line integral with scalar differential)**.** Let C be a (sufficiently nice) curve and $\boldsymbol{F}$ be a (sufficiently smooth) vector field. Then

$$\int_C \boldsymbol{F} \, \mathrm{d}s = \begin{pmatrix} \int_C F_1 \, \mathrm{d}s \\ \int_C F_2 \, \mathrm{d}s \\ \int_C F_3 \, \mathrm{d}s \end{pmatrix} \quad \text{i.e.} \quad \left(\int_C \boldsymbol{F} \, \mathrm{d}s \right)_i = \int_C F_i \, \mathrm{d}s. \tag{3.8}$$

(compare to Definition 2.8)

All the properties in Lemma 3.9 for line integrals of scalar fields hold for this first type of line integral of vector fields (because of this "component-wise" definition).

Work integrals

The second type of line integral over $\boldsymbol{F}$ is far more important in this course. It is written as

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{3.9}$$

Thus in the above integral, we take the component of force lying in the tangential direction along the curve (given by $|\boldsymbol{F}| \cos\theta$) and multiply by the arclength element, $|\mathrm{d}\boldsymbol{r}|$, and sum[1]. The result is a scalar. This scalar is identified with the physical concept of work. Formally, we define it as follows.

Definition 3.11 ("Work integral" of vector fields)**.** Let C be a (sufficiently nice) curve with parametrisation $\boldsymbol{r}(t)$ for $a \le t \le b$, and let $\boldsymbol{F}$ be a (sufficiently smooth)vector field. The *work integral* of the vector field $\boldsymbol{F}$ along C is defined by

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} := \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \, \mathrm{d}t. \tag{3.10}$$

[1]Recall $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = |\boldsymbol{F}||\mathrm{d}\boldsymbol{r}| \cos\theta$ where θ is the angle between the two vectors

The terminology "work integral" is slightly non-standard, but it will be useful to distinguish between the line integral (3.10) and other types of line integrals for vector fields such as (3.8).

Example 3.12 (Example of work integral)**.** Calculate the work integral (3.10) when $\boldsymbol{F} = (3xy, -5z, 10x)$ and the curve C is defined by the parametrisation $\boldsymbol{r}(t) = (t^2 + 1, 2t^2, t^3)$ for $t \in [1, 2]$.

Solution. Starting from the definition (3.10), we have

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &:= \int_1^2 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t, \\
&= \int_1^2 \begin{pmatrix} 3(t^2+1)2t^2 \\ -5t^3 \\ 10(t^2+1) \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 4t \\ 3t^2 \end{pmatrix} \mathrm{d}t \\
&= \int_1^2 \Big(12t^3(t^2+1) - 20t^4 + 30t^2(t^2+1) \Big) \mathrm{d}t, \\
&= \int_1^2 \Big(12t^5 + 10t^4 + 12t^3 + 30t^2 \Big) \mathrm{d}t = \ldots = 303.
\end{aligned}$$

Lemma 3.13 (Basic properties of work integrals)**.**

1. *Linearity:* if λ and μ are constants and $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields, then
$$\int_C (\lambda \boldsymbol{F} + \mu \boldsymbol{G}) \cdot \mathrm{d}\boldsymbol{r} = \lambda \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \mu \int_C \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{r}.$$
2. *Additivity w.r.t. curve:* if $C = C_1 \cup C_2$ then
$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}.$$
3. ***Non**-independence of direction:* with $-C$ defined by Definition 3.6,
$$\int_{-C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{3.11}$$

The first two properties follow directly from the definition (3.4); you will prove the third property in problem sets.

Chapter 4

The big theorem on conservative forces

As we mentioned at the start Chap. 3 and in Definition 2.20, many physical forces possess a special properties due to be conservative; indeed this allows great simplification of their study. Physically, a conservative force is typically defined as one where work performed by the force on a particle moving along a path, C, only depends on the chosen start- and end-points of C and not the particular path taken [cf. via (4.8)].

The mathematical definition of a conservative force or field, $\boldsymbol{F}$, in this course differs from the above. In Definition 2.20, we defined $\boldsymbol{F}$ as being conservative if it can be expressed as the gradient of a potential, i.e. $\boldsymbol{F} = \nabla\phi$. The point of this section is to establish under what conditions both these definitions are equivalent.

Let us start with the simplification that incurs if $\boldsymbol{F} = \nabla\phi$ in terms of the work integral.

Theorem 4.1 (Fundamental theorem of calculus for work integrals)**.** Let ϕ be a (sufficiently smooth) scalar field, and let C be a (sufficiently nice) curve with parametrisation $\boldsymbol{r}(t)$ for $a \leq t \leq b$. Then

$$\int_C \nabla\phi \cdot d\boldsymbol{r} = \phi(\boldsymbol{r}(b)) - \phi(\boldsymbol{r}(a)) \ . \tag{4.1}$$

Proof. By Definition 3.11,

$$\int_C \nabla\phi \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \nabla\phi(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t. \tag{4.2}$$

But by the chain rule and the definition of the gradient,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\boldsymbol{r}(t)\big) &= \frac{\partial\phi}{\partial x}\big(\boldsymbol{r}(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\phi}{\partial y}\big(\boldsymbol{r}(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\phi}{\partial z}\big(\boldsymbol{r}(t)\big)\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \nabla\phi\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t).\end{aligned}$$

Therefore,

$$\int_C \nabla\phi \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\boldsymbol{r}(t)\big)\,\mathrm{d}t,$$

and the result follows from the 1D fundamental theorem of calculus.

It turns out that

$$\exists\phi \text{ such that } \boldsymbol{F} = \nabla\phi \implies \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \text{ is independent of path}$$

so the first statement does not necessarily imply the second. In fact, the implication depends on the precise region where the force and potential functions are considered. We next define what it means to be a simply connected region.

Definition 4.2 (Simply connected)**.** A set $\Omega \subset \mathbb{R}^3$ is *simply connected* if (i) a path can be drawn between any two points in the Ω, and (ii) any closed curve $C \subset \Omega$ can be shrunk to a point in Ω without leaving Ω.

Finally we come to the main theorem on conservative forces.

Theorem 4.3 (The Big Theorem on conservative forces)**.** The following three statements are equivalent.

1. $\boldsymbol{F}$ is a conservative vector field on a simply connected domain Ω.
2. For every (sufficiently nice) closed curve $C \subset \Omega$,
$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0. \tag{4.3}$$
3. For any two (sufficiently nice) curves C_1 and C_2 that both have start point $\boldsymbol{x}_a$ and end point $\boldsymbol{x}_b$,
$$\int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \;=\; \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{4.4}$$

Proof. The proof proceeds in three parts:
(1 $\Rightarrow$ 2) Using both the fact that $\boldsymbol{F} = \nabla\phi$ for some potential ϕ and the fundamental theorem of calculus for work integrals (Thm. 4.1), we have

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}r = \oint_C \nabla\phi \cdot \mathrm{d}\boldsymbol{r} = \phi\big(\boldsymbol{r}(b)\big) - \phi\big(\boldsymbol{r}(a)\big).$$

However since C is closed, $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, so the above integral is zero.
(2 $\Rightarrow$ 3) The proof follows directly from considering two arbitrary contours C_1 and C_2 that share start and end-points, and then constructing a closed contour from the two individual elements.
(3 $\Rightarrow$ 1) The proof begins by defining

$$\phi(\boldsymbol{x}) := \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{4.5}$$

where C is any curve with start point $\mathbf{0}$ and end point $\boldsymbol{x}$, and verifying that $\boldsymbol{F} = \nabla\phi$. The proofs of 2 $\Rightarrow$ 3 and 3 $\Rightarrow$ 1 will be done as part of the Problem Set.

4.1 Conservation of energy

Work is define to be the product of displacement times force (oriented along the direction of displacement), but what is it exactly? (It may be further confusing as 'work' and 'energy' are often used interchangeably). By Newton's second law, force is mass times acceleration. For simplicity, consider the case that the force is aligned with the motion. Work is then equal to[1]

$$W = F \cdot \Delta s = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right),$$

[1]Note $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv^2) = ma$ where a is acceleration. Furthermore by elementary physics, for uniform acceleration, $\Delta v^2 = 2a\Delta s$.

where m is the mass. The bracketed term is the kinetic energy. Hence work is equal to change in kinetic energy and we see that the units of work are the same as the units for energy (in Joules, J).

But then what are the units of the potential, ϕ? Since ϕ is related to $\boldsymbol{F}$ by $\boldsymbol{F} = \nabla\phi$ then the units must be related by,

$$[\boldsymbol{F}] = \frac{[\phi]}{[x]} \implies [\phi] = \mathrm{N} \cdot \mathrm{m} \ \text{(Newton} \cdot \text{metre)}.$$

Alternatively, since $[\boldsymbol{F}] = \mathrm{J} \cdot \mathrm{m}^{-1}$ then the units of ϕ are also Joules.

Example 4.4 (Work equals change in kinetic energy)**.** Let $\boldsymbol{r}(t)$ be the position vector of a particle of mass m moving under a force $\boldsymbol{F}$. Define the curve C to be the particle's path between $t = a$ and $t = b$. Show that the work integral

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \tag{4.6}$$

is equal to the change in kinetic energy over the timespan of the motion.

Solution. The particle velocity is given by $\boldsymbol{r}'(t)$ and acceleration is $\boldsymbol{r}''(t)$. By Newton's 2nd law. force = mass × acceleration, and thus

$$\boldsymbol{F}(\boldsymbol{r}(t)) = m\boldsymbol{r}''(t). \tag{4.7}$$

The work done in moving our particle along C is

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{4.8}$$

Using (4.7) and the definition of work integrals (3.10), (4.8) becomes

$$W = m\int_C \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \cdot \mathrm{d}\boldsymbol{r} = m\int_a^b \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}(t) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t.$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = 2\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t},$$

so

$$\begin{aligned} W &= \frac{1}{2}m\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\boldsymbol{r}'(t)\right|^2\right)\mathrm{d}t \\ &= \frac{1}{2}m\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}(b)\right|^2 - \frac{1}{2}m\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}(a)\right|^2, \end{aligned}$$

which is the change in kinetic energy of the particle. In other words,

work done = change in kinetic energy from $t = a$ to $t = b$.

Example 4.5 (Conservation of energy)**.**

If $\boldsymbol{F}$ is conservative, i.e. $\boldsymbol{F} = \nabla\phi$ for some potential ϕ, then by Theorem 4.1

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C \nabla\phi \cdot d\boldsymbol{r} = \phi(\boldsymbol{r}(b)) - \phi(\boldsymbol{r}(a)).$$

and thus

work done = change in potential energy from $t = a$ to $t = b$.

We therefore have that, for motion under a conservative force, the change in potential energy equals the change in kinetic energy, i.e. conservation of energy.

Chapter 5

How to parameterise a surface

We have studied the computation of line integrals, and the next step is to study integrals defined over a surface. Before we do that, however, we must first discuss how surfaces are parameterised. Different applications require different representations. There are three main representations discussed below.

Definition 5.1 (Representations of surfaces in $\mathbb{R}^3$)**.**
1. *Explicit representation:* here given values of (x, y), a 'height' of the surface is given explicitly.

$$\big\{\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3 : \ z = f(x, y)\big\}. \tag{5.1}$$

2. *Implicit representation:* here we consider values of (x, y, z) that satisfy some implicit relation.

$$\big\{\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3 : \ F(\boldsymbol{x}) = F(x, y, z) = 0\big\}. \tag{5.2}$$

Note that given the explicit representation $z = f(x, y)$, an implicit representation is given by $F(x, y, z) := z - f(x, y)$.

3. *Parametric representation:* here consider values of x, y, and z given as functions of parameters u and v.

Note that implicit and parametric representations are not unique. For example the unit circle can be parameterised with $\boldsymbol{r} = (\cos 2\theta, \sin 2\theta)$ for $\theta \in [0, \pi)$. What are some other ways to parameterise the circle?

$$\big\{\boldsymbol{r} \equiv \boldsymbol{r}(u, v) = \big(x(u, v),\, y(u, v),\, z(u, v)\big) \text{ for } (u, v) \in D \subseteq \mathbb{R}^2\big\}. \tag{5.3}$$

Example 5.2 (Example of different representations)**.**
1. Explicit representation of a hemisphere of radius a:

$$\big\{(x, y, z) : \quad z = \sqrt{a^2 - x^2 - y^2}\big\}.$$

2. Implicit representation of the sphere of radius a:

$$\big\{(x, y, z) : \quad x^2 + y^2 + z^2 - a^2 = 0\big\}.$$

3. Parametric representation of the sphere of radius a:

$$\boldsymbol{r} = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi) \quad \text{for} \quad 0 \le \theta < 2\pi, \quad 0 \le \phi < \pi. \tag{5.4}$$

The next step is for us to characterise the normal (and hence the orientation) of a general three-dimensional surface. If the surface is a plane, then this is most straightforward. Recall some results about planes.

Lemma 5.3 (Vector equation of a plane)**.**
(a) The equation of a plane in $\mathbb{R}^3$ is

$$\boldsymbol{r} = \boldsymbol{r}_0 + \lambda \boldsymbol{v}_1 + \mu \boldsymbol{v}_2, \quad \lambda, \mu \in \mathbb{R}, \tag{5.5}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are any two non-parallel vectors in the plane.
(b) A unit normal to the plane (5.5) is

$$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{|\boldsymbol{v}_1 \times \boldsymbol{v}_2|}. \tag{5.6}$$

For a general surface, S, the normal will depend on the orientation of the surface at each particular point. The idea to developing a formula for $\widehat{\boldsymbol{n}}(\boldsymbol{x})$ is to consider the planar normal for an infinitesimal surface patch.

Lemma 5.4 (Finding normals to a surface)**.**
1. *Parametric representation:* suppose that the surface S has the parametric representation (5.3). A unit normal vector at $\boldsymbol{r}(u_0, v_0) \in S$ is then given by

$$\widehat{\boldsymbol{n}}\big(\boldsymbol{r}(u_0, v_0)\big) = \left.\frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}\right|_{(u_0, v_0)}. \tag{5.7}$$

where we have used subscripts for partial derivatives.

2. *Implicit representation:* suppose the surface S has the implicit representation (5.2). A unit normal vector at $\boldsymbol{x} \in S$ is then given by

$$\widehat{\boldsymbol{n}}(\boldsymbol{x}) = \frac{\nabla F(\boldsymbol{x})}{|\nabla F(\boldsymbol{x})|}. \tag{5.8}$$

Proof. Point 2 follows immediately from Corollary 2.18.

To prove Point 1, note that $\boldsymbol{r}(u, v_0)$ (for varying u) and $\boldsymbol{r}(u_0, v)$ (for varying v) are both curves in S passing through the point $\boldsymbol{r}(u_0, v_0)$. Thus, by Lemma 3.3, $\boldsymbol{r}_u(u_0, v_0)$ and $\boldsymbol{r}_v(u_0, v_0)$ are both vectors tangent to S at the point $\boldsymbol{r}(u_0, v_0)$. Hence their cross product (normalized) yields a unit normal. This result follows after applying Part (b) of Lemma 5.3.

Remark. Lemmas 5.3 and 5.4 show that in order for the parametric representation (5.3) to describe a genuinely 2-d surface, we need $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ not to be parallel, which is equivalent to $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$. This is a degeneracy condition on the surface.

Definition 5.5 (Orientable)**.** Let S be a smooth surface. At a point $P \in S$, choose one of the 2 unit normals $\widehat{\boldsymbol{n}}$ to be the *outward-pointing* unit normal vector. The surface S is *orientable* if the outward-pointing direction at P can then be continued in a unique and continuous way to the entire surface S.

Although one normally denotes unit vectors by a hat ($\widehat{\cdot}$), we will sometimes use $\boldsymbol{n}$ (without a hat) to denote the outwards-pointing unit normal to an orientable surface.

Chapter 6

The surface and flux integral

Definition 6.1 (Definition of a surface integral as a Riemann sum)**.** Given a (sufficiently smooth) surface $S \subset \mathbb{R}^3$, subdivide S into N subsurfaces $S_1, \ldots, S_N$ with surface areas $\delta S_1, \ldots, \delta S_N$. In each of the subsurfaces choose an arbitrary point $\boldsymbol{x}_i$. Then, for any scalar function f on S,

$$\iiint_S f \,\mathrm{d}S = \iiint_S f(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x}) := \lim_{N\to\infty} \sum_{i=1}^{N} f(\boldsymbol{x}_i)\, \delta S_i, \tag{6.1}$$

provided that this limit exists and is independent both of the choice of $\boldsymbol{x}_i$ and of the particular subdivision.

In practice we compute surface integrals using the parametric representation.

Theorem 6.2 (Surface integrals via the parametric representation)**.** Let S be a (sufficiently smooth) surface with parametric representation (5.3). Then

$$\iiint_S f(\boldsymbol{x}) \,\mathrm{d}S(\boldsymbol{x}) = \iint_D f\big(\boldsymbol{r}(u,v)\big) \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| \mathrm{d}u \,\mathrm{d}v, \tag{6.2}$$

i.e.

$$\mathrm{d}S = \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| \mathrm{d}u \,\mathrm{d}v; \tag{6.3}$$

we call $\mathrm{d}S$ the "surface element".

Proof. The main step in proving the theorem is to develop the correct expression for the area of a surface patch, $\mathrm{d}S$. Let δS be the parallelogram with corners $\boldsymbol{r}(u_0, v_0)$, $\boldsymbol{r}(u_0 + \delta u, v_0)$, $\boldsymbol{r}(u_0, v_0 + \delta v)$, and $\boldsymbol{r}(u_0 + \delta u, v_0 + \delta v)$. By Taylor's theorem, for δu small,

$$\boldsymbol{r}(u_0 + \delta u, v_0) = \boldsymbol{r}(u_0, v_0) + \frac{\partial \boldsymbol{r}}{\partial u}(u_0, v_0)\, \delta u + O(\delta u)^2,$$

and similarly for $\boldsymbol{r}(u_0, v_0 + \delta v)$.

The sides of the parallelogram are generated by the two vectors $\boldsymbol{r}(u_0 + \delta u, v_0) - \boldsymbol{r}(u_0, v_0)$ and $\boldsymbol{r}(u_0, v_0 + \delta v) - \boldsymbol{r}(u_0, v_0)$, and hence the area given by the magnitude of the cross product. Thus

$$\delta S \approx \left|\frac{\partial \boldsymbol{r}}{\partial u}(u_0, v_0) \times \frac{\partial \boldsymbol{r}}{\partial v}(u_0, v_0)\right| \delta u\, \delta v.$$

Using this in (6.1) and taking the limits of δu, $\delta v \to 0$ we find (6.2).

Remark 6.3 (Connection with the Jacobian)**.** It is important to note that Theorem 6.2, which demonstrates how to convert a surface element,

$$\mathrm{d}S = |\boldsymbol{r}_u \times \boldsymbol{r}_v| \,\mathrm{d}u \,\mathrm{d}v, \tag{6.4}$$

is in fact the same result and proof for conversion of coordinate systems via the Jacobian. Consider a mapping of a region R_{uv} in the (u, v) plane to a region R in the (x, y) plane. Consider $\boldsymbol{r} = \boldsymbol{r}(x(u, v),\, y(u, v), 0)$. You can verify that

$$|\boldsymbol{r}_u \times \boldsymbol{r}_v| = |(0, 0, x_u y_v - x_v y_u)| = \frac{\partial(x, y)}{\partial(u, v)}. \tag{6.5}$$

In other words, in 2D, the cross-product term is nothing more than the Jacobian factor which allows for conversion between coordinate systems. For example, using the polar coordinate mapping of,

$$\boldsymbol{r}(r, \theta) = (r\cos\theta, r\sin\theta, 0),$$

we find that

$$\mathrm{d}S = r \,\mathrm{d}r \,\mathrm{d}\theta,$$

which proves the familiar Jacobian for polar coordinates.

Example 6.4 (Surface area of a sphere)**.** In your problem set, you will use (6.3) to show that when S is the sphere of radius a, $\{\boldsymbol{x} : x^2 + y^2 + z^2 = a^2\}$,

$$\mathrm{d}S = a^2 \sin\phi \,\mathrm{d}\theta \,\mathrm{d}\phi, \tag{6.6}$$

where we are using the parametric representation (5.4) (i.e. spherical polar coordinates). Therefore, the surface area of this sphere equals

$$\iint_S \mathrm{d}S = a^2 \int_0^{\pi} \int_0^{2\pi} \sin\phi \,\mathrm{d}\theta \,\mathrm{d}\phi = 2\pi a^2 \int_0^{\pi} \sin\phi \,\mathrm{d}\phi = 4\pi a^2.$$

Lemma 6.5 (Surface element for explicit parametrisation)**.** If S is given by the explicit parametrisation (5.1) (i.e. $z = f(x, y)$) then

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \,\mathrm{d}x \,\mathrm{d}y \tag{6.7}$$

Idea for the proof of Lemma 6.5: Convert the explicit representation into a parametric representation by writing $\boldsymbol{r} = (x, y, f(x, y))$, and use the formula (6.3).

Proof. See Problem Set.

Definition 6.6 (Flux integral)**.** Let S be a (sufficiently nice) orientable surface with OPUNV $\boldsymbol{n}$, and let $\boldsymbol{F}$ be a (sufficiently smooth) vector field.

The *flux integral* of the vector field $\boldsymbol{F}$ on S is defined by

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} := \iint_S \boldsymbol{F} \cdot \boldsymbol{n} \,\mathrm{d}S; \tag{6.8}$$

i.e. $\mathrm{d}\boldsymbol{S} := \boldsymbol{n} \,\mathrm{d}S$.

The terminology "flux integral" is slightly non-standard.

Example 6.7 (Example of flux integral)**.** Find $\int_S \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S}$ where S is the sphere of radius a centred at the origin.

Solution. From the equation for $\boldsymbol{n}$ (5.8) and Example 2.19 we have $\boldsymbol{n} = \widehat{\boldsymbol{x}}$, and so

$$\begin{aligned}
\iint_S \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S} &:= \int_S \boldsymbol{x} \cdot \widehat{\boldsymbol{x}}\, \mathrm{d}S \quad \text{by the Definition of the flux integral (6.8)}, \\
&= \iint_S |\boldsymbol{x}| \mathrm{d}S \quad \text{since } \boldsymbol{x} \cdot \widehat{\boldsymbol{x}} = |\boldsymbol{x}|, \\
&= a \iint_S \mathrm{d}S \quad \text{since } |\boldsymbol{x}| = a \text{ on } S, \\
&= a \cdot 4\pi a^2 \quad \text{using Example 6.4}, \\
&= 4\pi a^3.
\end{aligned}$$

Lemma 6.8 (Physical interpretation of the flux integral of a vector field)**.** Suppose $\boldsymbol{F}$ is the velocity field of some fluid (i.e. the fluid's velocity at $\boldsymbol{x} \in \mathbb{R}^3$ is given by the vector $\boldsymbol{F}(\boldsymbol{x})$). Then $\int_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$ is the total volume of fluid that passes through S (in the "outward" direction) per unit of time.

Proof. Consider a small patch of surface δS.

Note that $\boldsymbol{F} \cdot \boldsymbol{n}$ is the component of the velocity in the normal direction. Since distance equals speed $\times$ time, $\boldsymbol{F} \cdot \boldsymbol{n}\, \delta t$ is the distance travelled by the exiting fluid in time δt.

Since volume equals distance $\times$ area, $\boldsymbol{F} \cdot \boldsymbol{n}\, \delta t\, \delta S$ is the volume occupied by the exiting fluid in time δt. Dividing by δt (and hence for a per-unit-time quantity) and summing over all patches δS we get the result.

Chapter 7

Divergence and curl

7.1 Divergence

Definition 7.1 (Divergence)**.** Let $\Omega \subset \mathbb{R}^3$ and let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a differentiable vector field. The *divergence* of $\boldsymbol{F} = (F_1, F_2, F_3)$ on Ω is defined by

$$\nabla \cdot \boldsymbol{F} := \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{7.1}$$

In index notation, this is simply written as

$$\nabla \cdot \boldsymbol{F} = \partial_i F_i. \tag{7.2}$$

Note that:

- $\boldsymbol{F}$ is a vector field, $\nabla \cdot \boldsymbol{F}$ is a scalar field.
- Sometimes people write $\operatorname{div} \boldsymbol{F}$ for $\nabla \cdot \boldsymbol{F}$.
- If we think of ∇ as $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, then $\nabla \cdot \boldsymbol{F}$ is the analogue of the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ with $\boldsymbol{a} = \nabla$ and $\boldsymbol{b} = \boldsymbol{F}$. However, whereas $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b}$
$$\boldsymbol{F} \cdot \nabla = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z} \neq \nabla \cdot \boldsymbol{F}$$
(this shows that $\boldsymbol{F} \cdot \nabla$ is actually a differential operator).

Example 7.2. Find $\nabla \cdot \boldsymbol{x}$ and $\nabla \cdot (-y, x, 0)$.

Solution.

$$\nabla \cdot \boldsymbol{x} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,$$

and

$$\nabla \cdot \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} + \frac{\partial(0)}{\partial z} = 0.$$

The divergence, div, is a linear operator and satisfies the usual properties of addition and scalar multiplication.

Lemma 7.3 (Basic properties of the divergence)**.** If $\boldsymbol{F}$ and $\boldsymbol{G}$ are differentiable vector fields, ϕ is a differentiable scalar field, and λ and μ are constants, then

$$\nabla \cdot (\lambda \boldsymbol{F} + \mu \boldsymbol{G}) = \lambda \nabla \cdot \boldsymbol{F} + \mu \nabla \cdot \boldsymbol{G}, \tag{7.3}$$

and

$$\nabla \cdot (\phi \boldsymbol{F}) = (\nabla \phi) \cdot \boldsymbol{F} + \phi (\nabla \cdot \boldsymbol{F}). \tag{7.4}$$

Idea for the proof of Lemma 7.7. For (7.3), use the definition of the divergence (7.1) and the linearity of differentiation. For (7.4), use the definition of the divergence (7.1) and the product rule.

Definition 7.4 (Solenoidal/incompressible)**.** A vector field $\boldsymbol{F}$ is called *solenoidal* or *incompressible* if $\nabla \cdot \boldsymbol{F} = 0$.

7.2 Curl

Definition 7.5 (Curl)**.** Let $\Omega \subset \mathbb{R}^3$ and let $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a differentiable vector field. The *curl* of $\boldsymbol{F} = (F_1, F_2, F_3)$ on Ω is defined by

$$\begin{aligned} \nabla \times \boldsymbol{F} &:= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}. \end{aligned} \tag{7.5}$$

In index form, this can be succinctly written as

$$(\nabla \times \boldsymbol{F})_i = \epsilon_{ijk}\partial_j \boldsymbol{F}_k. \tag{7.6}$$

Note:

- $\boldsymbol{F}$ is a vector field, $\nabla \times \boldsymbol{F}$ is a vector field.
- Sometimes people write $\operatorname{curl} \boldsymbol{F}$ for $\nabla \times \boldsymbol{F}$.
- If we think of ∇ as $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, then $\nabla \times \boldsymbol{F}$ is the analogue of the vector product $\boldsymbol{a} \times \boldsymbol{b}$ with $\boldsymbol{a} = \nabla$ and $\boldsymbol{b} = \boldsymbol{F}$. However, whereas $\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{F} \times \nabla \neq -\nabla \times \boldsymbol{F}$ (indeed, the right-hand side is a differential operator).

Example 7.6 (Examples of finding the curl)**.** Find $\nabla \times \boldsymbol{x}$ and $\nabla \times (-y, x, 0)$.

Solution.

$$\nabla \times \boldsymbol{x} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\nabla \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Lemma 7.7 (Basic properties of the curl)**.** If $\boldsymbol{F}$ and $\boldsymbol{G}$ are differentiable vector fields, ϕ is a differentiable scalar field, and λ and μ are constants, then

$$\nabla \times (\lambda \boldsymbol{F} + \mu \boldsymbol{G}) = \lambda \nabla \times \boldsymbol{F} + \mu \nabla \times \boldsymbol{G}, \tag{7.7}$$

and

$$\nabla \times (\phi \boldsymbol{F}) = (\nabla \phi) \times \boldsymbol{F} + \phi(\nabla \times \boldsymbol{F}). \tag{7.8}$$

Idea for the proof of Lemma 7.7. For (7.7), use the definition of the curl (7.5) and the linearity of differentiation. For (7.8), use the definition of the curl (7.5) and the product rule.

Definition 7.8 (Irrotational)**.** A vector field $\boldsymbol{F}$ is called *irrotational* if $\nabla \times \boldsymbol{F} = \boldsymbol{0}$.

Remark (Divergence measures expansion of a vector field, curl measures rotation of a vector field)**.** This is indicated by Examples 7.2 and 7.6.

For instance, note that if $\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$, then $\boldsymbol{x}$ moves in a circle in a plane with normal $\boldsymbol{\omega}$. The vector $\boldsymbol{\omega}$ is then called the angular velocity of $\boldsymbol{x}$. One can show that $\nabla \times (\boldsymbol{\omega} \times \boldsymbol{x}) = 2\boldsymbol{\omega}$. Thus, the curl of the velocity vector equals twice the angular velocity.

7.3 Second derivatives

In association with the differential operators ∇ (grad), $\nabla\cdot$ (div), and $\nabla\times$ (curl) there are now different types of possible second derivatives.

div	$\nabla \cdot \boldsymbol{F}$
grad	$\nabla\phi$
curl	$\nabla \times \boldsymbol{F}$
div grad	$\nabla \cdot \nabla\phi$
curl grad	$\nabla \times \nabla\phi$
grad div	$\nabla(\nabla \cdot \boldsymbol{F})$
div curl	$\nabla \cdot (\nabla \times \boldsymbol{F})$
curl curl	$\nabla \times (\nabla \times \boldsymbol{F})$

It is important to consider the sensibility of different combinations. For example, $\nabla(\nabla \times \boldsymbol{F})$, i.e. grad(curl $\boldsymbol{F}$), does not make sense because the gradient can only act on a scalar. Similarly, $\nabla \times (\nabla \cdot \boldsymbol{F})$, i.e. curl(div $\boldsymbol{F}$), is also incorrect as the curl can only act on a vector.

Definition 7.9 (Laplace operator)**.** For a scalar field f, we define the Laplacian of f as

$$\nabla^2 f := \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

we call ∇^2 the *Laplace operator* or *Laplacian*. The Laplacian is also sometimes written as Δ. For a vector field $\boldsymbol{F}$, the operator is similarly defined by

$$\nabla^2 \boldsymbol{F} := \begin{pmatrix} \nabla^2 F_1 \\ \nabla^2 F_2 \\ \nabla^2 F_3 \end{pmatrix};$$

The equation

$$\nabla^2 f = 0,$$

is known as Laplace's equation and is one of the essential partial differential equations to be studied later.

Lemma 7.10 (Properties of certain second derivatives)**.** Provided that f is sufficiently nice enough for its second partial derivatives to be symmetric, i.e. $f_{xy} = f_{yx}$ etc., and provided that $\boldsymbol{F}$ is sufficiently nice enough for the second partial derivatives of each of its components to be symmetric, then

1. $\nabla \times (\nabla f) = \boldsymbol{0}$.

2. $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$.

3. $\nabla(\nabla \cdot \boldsymbol{F}) - \nabla \times (\nabla \times \boldsymbol{F}) = \nabla^2 \boldsymbol{F}$.

Proof of Part 1 of Lemma 7.10. We'll prove Part 1 here, and the proofs of Parts 2 and 3 are on Problem Set. By the definitions of gradient and curl, and using the abbreviations $\partial_x := \partial/\partial_x$ and $f_x := \partial f/\partial x$, we have

$$\nabla \times (\nabla f) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\boldsymbol{i} - (f_{zx} - f_{xz})\boldsymbol{j} + (f_{yx} - f_{xy})\boldsymbol{k}$$

which is zero by the hypothesis about the symmetry of the second order partial derivatives.

Chapter 8

The divergence theorem and Green's theorem

In this chapter, we shall study one of the most important vector theorems. The Divergence theorem states that

$$\iiint_\Omega \nabla \cdot \boldsymbol{F} \, \mathrm{d}V = \iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}. \tag{8.1}$$

It is essentially a conservation law, indicating that the flux of a certain quantity through a surface, S (the right hand-side) is equal to a certain local volumetric quantity (the divergence of $\boldsymbol{F}$). We prove the Divergence theorem in this chapter, and discuss its various applications and corollaries (one of which is the well known Green's theorem).

8.1 Geometric definitions

We first need to introduce some geometric definitions. Although the fundamental theorem of calculus in 1D takes place on an interval, the multidimensional analogues take place on sets with higher dimension and so we need to be more prescriptive about what type of sets are allowed.

Definition 8.1 (Domain)**.** A *domain* is an open, connected subset of $\mathbb{R}^3$.

Definition 8.2 (Bounded)**.** A domain $\Omega \subset \mathbb{R}^3$ is *bounded* if there exists an $R > 0$ such that $\Omega \subset B_R$, where $B_R := \{\boldsymbol{x} : |\boldsymbol{x}| < R\}$ (i.e. B_R is the ball with radius R and centre $\mathbf{0}$).

Definition 8.3 (Convex)**.** A domain Ω is *convex* if given two points $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in \Omega$, the line segment $\{t\boldsymbol{x}_1 + (1-t)\boldsymbol{x}_2 : t \in (0,1)\}$ is a subset of Ω.

Corollary 8.4 (Equivalent characterisation of convex domains)**.** A domain Ω is convex if every straight line intersects $\partial\Omega$ at two points at most.

Definition 8.5 (Open and closed surfaces)**.** A surface $S \subset \mathbb{R}^3$ is *open* if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \notin S$ there exists a curve from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ which does not cross S. A surface $S \subset \mathbb{R}^3$ is *closed*, if it is not open.

Note that this use of the words "open" and "closed" to describe surfaces is in a different sense to how "open" and "closed" are used to describe sets.

8.2 The divergence theorem

Theorem 8.6 (The divergence theorem)**.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with (sufficiently smooth) boundary $\partial\Omega$ and with outward-pointing unit normal vector $\boldsymbol{n}$. If $\boldsymbol{F}$ is a continuously differentiable vector field, then

$$\iiint_\Omega \nabla \cdot \boldsymbol{F}\,\mathrm{d}V = \iint_S \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S}, \tag{8.2}$$

where $\mathrm{d}\boldsymbol{S} := \boldsymbol{n}\,\mathrm{d}S$ and we have written $S = \partial\Omega$.

Proof. We only prove the Divergence Theorem for convex domains, though the proof extends with very little extra work to any region that can be decomposed into convex regions.

Let D be the projection of Ω onto the (x,y)–plane. Consider the line L through the point $(x,y,0)$ parallel to the z–axis. Since S is convex, L intersects S at two points $(x,y,f(x,y))$ and $(x,y,g(x,y))$, where $f(x,y)\le g(x,y)$ for all $(x,y)\in D$. In other words we can write

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 \,:\, (x,y)\in D, f(x,y)\le z\le g(x,y)\}. \tag{8.3}$$

We break the Divergence Theorem into the three component directions. Thus if we can prove that

$$\iiint_\Omega \frac{\partial F_1}{\partial x}\,\mathrm{d}V = \iint_{\partial\Omega} F_1\,\boldsymbol{i}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{8.4a}$$

$$\iiint_\Omega \frac{\partial F_2}{\partial y}\,\mathrm{d}V = \iint_{\partial\Omega} F_2\,\boldsymbol{j}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{8.4b}$$

$$\iiint_\Omega \frac{\partial F_3}{\partial z}\,\mathrm{d}V = \iint_{\partial\Omega} F_3\,\boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{8.4c}$$

then the sum of the three results establishes the Divergence Theorem.

Let us focus first on (8.4c). The others will follow from symmetry. On the left hand-side, we have

$$\begin{aligned}\iiint_\Omega \frac{\partial F_3}{\partial z}\,\mathrm{d}V &= \iint_D \left(\int_{f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}(x,y,z)\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y\\ &= \iint_D \Big(F_3\big(x,y,g(x,y)\big) - F_3\big(x,y,f(x,y)\big)\Big)\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{8.5}$$

Turning to the right hand-side of (8.4c), we split the surface into the upper and lower surfaces, writing

$$\iint_S F_3\,\boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S = \left(\iint_{S_1} + \iint_{S_2}\right) F_3\,\boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{8.6}$$

where

$$S^+ = \{(x,y,z)\in\mathbb{R}^3 : (x,y)\in D, z = g(x,y)\} \tag{8.7}$$

$$S^- = \{(x,y,z)\in\mathbb{R}^3 : (x,y)\in D, z = f(x,y)\}. \tag{8.8}$$

We write the surface elements using the parametric representations. From Theorem 6.2, we have

$$\text{on } S^+: \qquad \mathrm{d}\boldsymbol{S} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & g_x\\ 0 & 1 & g_y\end{vmatrix}\mathrm{d}x\,\mathrm{d}y = (-g_x,-g_y,1)\,\mathrm{d}x\,\mathrm{d}y \tag{8.9}$$

$$\text{on } S^-: \qquad \mathrm{d}\boldsymbol{S} = -\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix}\mathrm{d}x\,\mathrm{d}y = (f_x,f_y,-1)\,\mathrm{d}x\,\mathrm{d}y. \tag{8.10}$$

It thus follows that the surface integrals are

$$\iint_{S^+} F_3\,\boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S = \iint_D F_3\big(x,y,g(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \tag{8.11}$$

$$\iint_{S^-} F_3\,\boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S = -\iint_D F_3\big(x,y,f(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \tag{8.12}$$

Adding the two then yields the right hand-side of (8.5), and hence the z-component part of (8.4c) is proved. The proofs in the other two Cartesian directions are done similarly (and by symmetry must be valid). Hence we have shown

$$\iiint_\Omega \nabla \cdot \boldsymbol{F}\,\mathrm{d}V = \iint_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S}, \tag{8.13}$$

for any convex body.

Example 8.7. Verify that the divergence theorem holds when $\boldsymbol{F} = \boldsymbol{x}$ and Ω is the ball of radius a centred at the origin.

Solution. By Example 6.7, the surface integral, $\int_S \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = 4\pi a^3$. Also by Example 7.2, $\nabla\cdot\boldsymbol{F} = \nabla\cdot\boldsymbol{x} = 3$, and so the left-hand side of (8.13) equals $3\int_V \mathrm{d}V$. Using spherical polar coordinates (Definition 2.11),

$$\iiint_V \mathrm{d}V = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi} r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \ldots = \frac{4}{3}\pi a^3,$$

so $3\int_V \mathrm{d}V = 4\pi a^3$ as required.

8.3 Green's theorem in the plane

Below, recall that a **closed** set contains its boundary and a **simple closed curve** is one that has its endpoints meet but does not otherwise cross itself.

Green's theorem below is an application of the divergence theorem to planar areas and curves that bound such areas. For this theorem, it will be important that we establish the notion of a positively-oriented curve.

Definition 8.8 (Anticlockwise/positive orientation of boundary curves)**.** Let $D \subset \mathbb{R}^2$ be a bounded domain such that its boundary curve C is simple and sufficiently smooth. We say that C is *oriented in the anticlockwise or positive sense* if, for the given parameterization, $\boldsymbol{r}(t)$, $t \in [a,b]$, of C, as t increases, $\boldsymbol{r}(t)$ moves around C in the anticlockwise direction.

Theorem 8.9 (Green's Theorem in the plane—Divergence Theorem form)**.** Let $D \subset \mathbb{R}^2$ be a bounded domain such that its boundary curve C is simple, sufficiently smooth, and oriented in the anticlockwise/positive sense. Let $p(x,y)$ and $q(x,y)$ be differentiable in D. Then

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)\mathrm{d}A = \oint_C (p,q)\cdot\boldsymbol{n}\,\mathrm{d}s, \tag{8.14}$$

where $\boldsymbol{n}$ is the outward pointing unit normal to C in the (x,y) plane.

Proof. This is a corollary of the divergence theorem applied to a plane. The trick is to define a three-dimensional volume that reduces to the necessary planar version. Let $\boldsymbol{F} = (p(x,y), q(x,y), 0)$ and let us define the three-dimensional surface

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 | (x,y) \in D,\, z \in [0,1]\}.$$

In the following, we write $\tilde{\boldsymbol{n}}$ for the 3D normal associated with the divergence theorem [to be distinguished from $\boldsymbol{n}$, the 2D planar normal in the statement of eqn (8.16)]. The left side of the divergence theorem gives

$$\iiint_\Omega \nabla\cdot\boldsymbol{F}\,\mathrm{d}V = \iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)\mathrm{d}x\mathrm{d}y\left(\int_0^1 \mathrm{d}z\right),$$

which follows since the integrand does not depend on z. Thus by the divergence theorem,

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \iint_{\partial\Omega} \boldsymbol{F}\cdot\tilde{\boldsymbol{n}}\,\mathrm{d}S.$$

The surface $\partial\Omega$ is composed of the top ($z = 1$) and bottom ($z = 0$) sections. However, for these two sections, $\boldsymbol{F} \cdot \tilde{\boldsymbol{n}} = 0$ since the normal points either up or down, and $\boldsymbol{F}$ does not contain a $\boldsymbol{k}$ component. This leaves the 'band' given by

$$\{(x, y, z) \in \mathbb{R}^3 | (x, y) \in C,\, z \in [0, 1]\}.$$

On this band, notice that the outwards normal, $\tilde{\boldsymbol{n}}$, corresponds to the same normal as C, $\boldsymbol{n}$, but with a zero $\boldsymbol{k}$ component. Moreover on this section,

$$\mathrm{d}S = \mathrm{d}s\,\mathrm{d}z,$$

by consideration of the geometry (essentially an arclength segment $\mathrm{d}s$, of height $\mathrm{d}z$. Hence the surface integral is written as

$$\iint_{\partial\Omega} \boldsymbol{F} \cdot \tilde{\boldsymbol{n}}\,\mathrm{d}S = \int_0^1 \left\{ \int_C (p, q) \cdot \boldsymbol{n}\,\mathrm{d}s \right\} \mathrm{d}z = \int_C (p, q) \cdot \boldsymbol{n}\,\mathrm{d}s.$$

The last equality follows from the z-independence.

The divergence form of Green's Theorem in 8.9 is equivalent, but not exactly the same as the classical formulation of Green's theorem, which is written with the integral $\int_C (p, q) \cdot \boldsymbol{n}\,\mathrm{d}s$ according to a tangential derivative.

First, we remark the following:

Remark 8.10. By parameterizing C in the form $(x(s), y(s))$ where s is the arclength of C, increasing in the anticlockwise direction, one has the unit tangent $\boldsymbol{t}$ and outward unit normal, $\boldsymbol{n}$, given by

$$\boldsymbol{t} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s} \right), \qquad \boldsymbol{n} = \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s} \right). \tag{8.15}$$

Theorem 8.11 (Green's Theorem in the plane—Stokes' Theorem form)**.** Let $D \subset \mathbb{R}^2$ be a bounded domain such that its boundary curve C is simple, sufficiently smooth, and oriented in the positive sense. Suppose that $\boldsymbol{F} = (F_1, F_2) : D \to \mathbb{R}^2$ is a vector field in $\mathbb{R}^2$. Then

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{8.16}$$

Proof. This is a simple application and re-labeling of Theorem 8.9. Let $(p, q) = (F_2, -F_1)$. Then according to (8.14),

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \oint_C (F_2, -F_1) \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s} \right) \mathrm{d}s,$$

where we have used the expression for $\boldsymbol{n}$ in (8.15). Thus re-organizing the right hand-side yields

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \oint_C (F_1, F_2) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s} \right) \mathrm{d}s.$$

The integrand in the right hand-side is precisely $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ by Definition 3.11.

Corollary 8.12 (Separated differentials form of Green's Theorem)**.** Green's Theorem 8.11 can alternatively be written with the differentials separated:

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y = \oint_C F_1\,\mathrm{d}x + F_2\,\mathrm{d}y. \tag{8.17}$$

Proof. Take (8.16) and write $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \boldsymbol{F} \cdot \boldsymbol{t}\mathrm{d}s$ where $\boldsymbol{t}$ is the tangent vector (8.15).

Here is a simple application of Green's theorem.

Corollary 8.13 (Working out an area via a line integral)**.** Let $C \subset \mathbb{R}^2$ be a simple smooth closed curve that is oriented in the positive sense. The area enclosed by C is

$$\frac{1}{2}\oint_C (-y\boldsymbol{i} + x\boldsymbol{j}) \cdot \mathrm{d}\boldsymbol{r}.$$

Proof. Denote the area enclosed by C by D. Applying (8.16) with $F_1(x, y) = -y$ and $F_2(x, y) = x$, we have

$$2\iint_\Omega \mathrm{d}A = \oint_C (-y\boldsymbol{i} + x\boldsymbol{j}) \cdot \mathrm{d}\boldsymbol{r}.$$

Chapter 9

Stokes' theorem

Recall that the Divergence Theorem of the previous chapter relates surface quantities (the "flux", $\boldsymbol{F}\cdot\boldsymbol{n}$) to volumetric quantities (the "divergence", $\nabla\cdot\boldsymbol{F}$). The last theorem to be introduced in this chapter is Stokes' theorem, which relates surface quantities (the "curl", $\nabla\times\boldsymbol{F}$) to a line integral around a bounding curve:

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}.$$

Stokes' Theorem applies to the case of open and orientable surfaces (in essence, surfaces that possess a bounding curve).

Definition 9.1 ("Correspondingly orientated" surface and boundary curve)**.** Let S be an open, orientable surface with simple boundary curve C. Make a choice for "outward" and then let $\boldsymbol{n}$ be the outward-pointing unit normal on S. Imagine a person walking along the curve C (in the direction given by the parametrisation) with their head pointing in the direction of $\widehat{\boldsymbol{n}}$. S and C are said to be *correspondingly orientated* if the surface is to the left of the person.

In fact, we had already encountered a version of Stokes' Theorem previously—in the situation where the surface is a planar area in $\mathbb{R}^2$, Stokes' Theorem reduces to Green's Theorem in the plane in Theorem 8.11.

Theorem 9.2 (Stokes' theorem)**.** Let S be a sufficiently smooth, bounded, open, orientable surface in $\mathbb{R}^3$, with outward-pointing unit normal vector $\boldsymbol{n}$, and let C be its closed boundary curve. Let S and C be corresponding oriented (as in Definition 9.1). If $\boldsymbol{F}$ is a smooth vector field in a neighbourhood of S, then

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}, \tag{9.1}$$

where $\mathrm{d}\boldsymbol{S} := \boldsymbol{n}\,\mathrm{d}S$.

Proof. We will not prove the full Stokes' theorem, which applies for general surfaces S. However, we will prove the version where S can be parameterized by $\boldsymbol{r}(u,v)$, and where $(u,v)\in S_{uv}\subseteq\mathbb{R}^2$. More complicated surfaces can always be subsectioned so as to apply the simpler geometry.

Notice that if we can prove the version of Stokes' theorem for

$$\boldsymbol{F} = (F,0,0)$$

then similar proofs apply for $\boldsymbol{F}=(0,F,0)$ and $\boldsymbol{F}=(0,0,F)$, and by linearity, the full result follows.

Thus on the assumption $\boldsymbol{F}=(F,0,0)$, we seek to show that

$$\oint F\,\mathrm{d}x = \iint_S (0,F_z,-F_y)\cdot\mathrm{d}\boldsymbol{S}. \tag{9.2}$$

Under the parameterization $\boldsymbol{r}(u,v)$, the flux integral on the right then equals (by Theorem 6.2),

$$\begin{aligned}\iint_{S_{uv}} (0,F_z,-F_y)\cdot \mathrm{d}\boldsymbol{S} &= \iint_{S_{uv}} (0,F_z,-F_y)\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,\mathrm{d}u\mathrm{d}v \\ &= \iint_{S_{uv}} (0,F_z,-F_y)\cdot\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ x_u & y_u & z_u\\ x_v & y_v & z_v\end{vmatrix}\mathrm{d}u\mathrm{d}v \\ &= \iint_{S_{uv}} [F_z(z_u x_v - x_u z_v) - F_y(x_u y_v - y_u x_v)]\,\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{9.3}$$

We turn to the line integral on the left of (9.2). Since $x = x(u,v)$, then by the chain rule,

$$\oint_C F\,\mathrm{d}x = \int_{\partial S_{uv}} F(x_u\mathrm{d}u + x_v\mathrm{d}v).$$

Now we apply Green's Theorem in (8.17) to the right integral with the vector field (Fx_u, Fx_v). Hence this results in

$$\begin{aligned}\oint_C F\,\mathrm{d}x &= \iint_{S_{uv}} \left[\frac{\partial(Fx_v)}{\partial u} - \frac{\partial(Fx_u)}{\partial v}\right]\mathrm{d}u\mathrm{d}v \\ &= \iint_{S_{uv}} [F_u x_v - F_v x_u]\,\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{9.4}$$

Finally, the derivatives F_u and F_v are expanded via the chain rule, with

$$\begin{aligned}F_u &= F_x x_u + F_y y_u + F_z z_u \\ F_v &= F_x x_v + F_y y_v + F_z z_v.\end{aligned}$$

Once these identities are substituted into the right hand-side of (9.4), this returns precisely (9.3). Hence Stokes' theorem has been proven for the x-component argument. The y and z components follow similarly and the full theorem follows by linearity.

9.1 Applications of Stokes' theorem

Recall these three facts:

1. A conservative vector field $\boldsymbol{F}$ is one such that $\boldsymbol{F} = \nabla\phi$ for some potential ϕ (Definition 2.20);
2. The Big Theorem on conservative forces (Theorem 4.3) characterizes conservative vector fields in terms of certain properties of their line integrals; and
3. If $\nabla\times\boldsymbol{F} = \boldsymbol{0}$, then $\boldsymbol{F}$ is irrotational (Definition 7.8).

We now use Stokes' theorem to prove that $\boldsymbol{F}$ conservative $\iff$ $\boldsymbol{F}$ is irrotational, i.e., $\boldsymbol{F} = \nabla\phi \iff \nabla\times\boldsymbol{F} = \boldsymbol{0}$.

Theorem 9.3 ($\boldsymbol{F}$ conservative $\iff$ $\boldsymbol{F}$ is irrotational)**.** Let $\Omega\subset\mathbb{R}^3$ be a simply connected domain. $\boldsymbol{F}$ is conservative on Ω if and only if $\boldsymbol{F}$ is irrotational on Ω.

Proof. ($\Rightarrow$) Since $\boldsymbol{F}$ is conservative, we have that

$$\nabla\times\boldsymbol{F} = \nabla\times(\nabla\phi) = 0$$

which follows by the vector identity that curl-grad equals zero (Lemma 7.10).

($\Leftarrow$) We see to show that $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = 0$ for any closed curve $C\subset\Omega$, in which case Theorem 4.3 then implies that $\boldsymbol{F}$ is conservative.

Let C be any curve in Ω, and let S be a correspondingly oriented surface in Ω whose boundary curve is C. Note that since the domain is simply connected, we can always find such a surface, S. By assumption, $\nabla \times \boldsymbol{F} = 0$ for all $\boldsymbol{x} \in \Omega$, and thus it follows from Stokes' theorem that

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} \, \mathrm{d}S = 0.$$

This completes both directions of the proof.

Chapter 10

An introduction to partial differential equations

A partial differential equation (PDE) is a differential equation that involves a multivariate function, say $u(x, y, t)$, and its partial derivatives. We will mainly study the three canonical PDEs, which include the heat or diffusion equation,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \tag{10.1}$$

for a function $u(x, t)$ where κ is constant. This equation describes the diffusion of heat, as measured by u, given at a position x and time t.

We also have the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{10.2}$$

for a function $u(x, t)$ where c is constant. This equation describes the transverse displacement, u, of a stretched string at position x and time t.

Finally we have Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{10.3}$$

for a function $u = u(x, y)$. This equation comes from a variety of sources, but physically, it can represent the steady-state (or long-time) solution of a heat equation for a two-dimensional system. In such a scenario, u represents the temperature of a region in the (x, y) plane.

In order to develop solutions to each of these three examples of PDEs, each equation must be accompanied by specification of an appropriate domain where the equations are solved, as well as appropriate boundary or initial conditions. In many cases, what constitutes an 'appropriate' condition is informed by the physics.

Before beginning, note the following lemma, which we shall not prove.

Lemma 10.1 (The Bump Lemma)**.** Let $f(\boldsymbol{x})$ be a sufficiently smooth function defined on $\Omega \subseteq \mathbb{R}^n$. Suppose it is the case that

$$\int_V f(\boldsymbol{x}) \, \mathrm{d}V = 0$$

for all $V \subseteq \Omega$. Then

$$f(x) \equiv 0$$

in Ω.

Proof. (Idea only) The basic idea in proving the Bump Lemma is the following: assume that it is not the case that $f(x) = 0$ everywhere. Consider some $x = x_0$ where $f(x_0) \neq 0$. Then form a small interval

around this point, which invalidates the original assumption that the integral is zero for all possible sub-intervals.

Note that in particular the application of this theorem to $\mathbb{R}$. Consider a function defined on $[0, L]$. If we know that for any interval $[a, b]$ lying in $[0, L]$,

$$\int_a^b f(x)\,\mathrm{d}x = 0,$$

then necessarily $f(x) \equiv 0$ in $[0, L]$.

10.1 A physical derivation of the heat equation

Consider a straight rigid metal rod lying on the x-axis. The lateral surface is insulated to prevent heat loss. Let ρ be the mass density per unit length, c be the specific heat, and $T(x, t)$ be the temperature. Thus the amount of internal energy per unit length will be ρcT. Moreover, let us define the heat flux, $q(x, t)$, to be the heat flux moving in the positive x direction. You may want to examine the definition of standard units.[1]

Consider an interval $x \in [a, a + h]$ of the rod. Firstly, the internal heat energy in this segment is given by

$$\text{internal energy} = \int_a^{a+h} \rho cT(x, t)\,\mathrm{d}x. \tag{10.4}$$

There may also be heat flux (or movement of heat) through the boundaries at $x = a$ and $x = a + h$. Thus

$$\text{net heat flux in} = q(a, t) - q(a + h, t). \tag{10.5}$$

Notice that if $q(a + h, t) > 0$ then heat is flowing *out*. This explains the negative sign. By conservation of energy, the rate of change of internal energy must equal the net heat flux into the system. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{a+h} \rho cT(x, t)\,\mathrm{d}x = -\Big[q(a + h, t) - q(a, t)\Big]. \tag{10.6}$$

Since the limits of integration are fixed, we may pass $\frac{\mathrm{d}}{\mathrm{d}t}$ through the integral. Moreover, the right hand-side is re-written in terms of the fundamental theorem of Calculus. This gives,

$$\int_a^{a+h} \rho c\frac{\partial T}{\partial t}\,\mathrm{d}x = -\int_a^{a+h} \frac{\partial q}{\partial x}\,\mathrm{d}x. \tag{10.7}$$

Now, because the above integrals must hold for all possible values of a and $a + h$, then by Lemma 10.1, it must be the case that the integrands, themselves, are equal. Hence we conclude that

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}. \tag{10.8}$$

The problem is that we must relate the heat flux, q, to the temperature, T. This is a *constitutive relation* and is known as Fourier's law. Fourier's law states that the flux is proportional to the negative gradient in temperature, or

$$q(x, t) = -k\frac{\partial T}{\partial x}, \tag{10.9}$$

where k is the thermal conductivity of the material. The negative sign reflects the fact that heat flows from high temperatures to low temperatures. With Fourier's Law, the equation (10.8) is now

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}, \tag{10.10}$$

where $\kappa = k/(\rho c)$ is the thermal diffusivity.

[1] The standard dimensional units of the quantities are: (i) density, $[\rho] = \mathrm{kg/m}$; (ii) specific heat, $[c] = \mathrm{J/(kg\ K)}$; (iii) temperature, $[T] = \mathrm{K}$. Square brackets are used to denote dimensions, and here kg denotes kilograms, m for metres, J for joules, and K for Kelvins.

10.2 A physical derivation of the wave equation

Consider a flexible string stretched to a tension T, with mass density ρ. The string is initially at rest along the x-axis in the (x, y)-plane, and $y = y(x)$ denotes the distance of the string from the axis. We assume that the string assumes only small vibrations, so e.g. y^2 and y_x^2 are small, and that the motion is entirely traverse. Gravity and air resistance is ignored.

Consider a small segment of string, $x \in [x_1, x_2]$. Let θ_1 and θ_2 be the angle the string makes to the horizontal at $x = x_1$ and x_2. Similarly, let T_1 and T_2 be the tension (the force in the tangential component) in the string at these two points. We balance forces using Newton's second law. This gives

$$\text{horizontal forces:} \qquad T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0, \tag{10.11}$$

$$\text{vertical forces:} \qquad T_2 \sin\theta_2 - T_1 \sin\theta_1 = (\rho \Delta x) y_{tt}(x_0, t), \tag{10.12}$$

where the acceleration, y_{tt} is measured at a point $x_0 \in [x_1, x_2]$.

Note that if y_x is small, then θ_1 and θ_2 are both small. By the approximation $\cos\theta \sim 1$ for θ small, then $T_2 \cos\theta_2 \sim T_2$ and $T_1 \cos\theta_1 \sim T_1$. By the force balance in the horizontal direction, we conclude $T_1 = T_2$ and hence the tension is approximately constant over this inverval.

Turning now to the vertical equation (10.12), note that for small angles (or alternatively, by examining the usual trignometric triangle),

$$\sin\theta \sim \frac{\partial y}{\partial x}. \tag{10.13}$$

Hence writing $\sin\theta_1 \sim y_x(x_1, t)$ and $\sin\theta_2 \sim y_x(x_2, t)$, and setting $T_1 = T_2$, we have for the left hand-side of (10.12),

$$\frac{\sin\theta_2 - \sin\theta_1}{\Delta x} \sim \frac{y_x(x_1 + \Delta x) - y_x(x_0)}{\Delta x} = y_{xx}(a, t), \tag{10.14}$$

at some point $a \in [x_1, x_2]$ by the Mean Value Theorem. Finally, letting $\Delta x \to 0$, and assuming that $y_x x$ and $y_t t$ are continuous, we conclude that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{10.15}$$

where $c = \sqrt{T/\rho}$.

10.3 A physical derivation of Laplace's equation

Let us consider the extension of the physical derivation of the heat equation of Sec.10.1 to the temperature distribution of a two-dimensional surface, $(x, y) \in D \subseteq \mathbb{R}^2$. Now the temperature is specified by $T(x, y)$. It can be shown that application of conservation of energy then implies that

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \tag{10.16}$$

where κ is the thermal diffusivity. This is the heat equation in 2D and reduces to (10.10) when T is independent of y.

Now consider a system where, after a long time, $t \to \infty$, the surface being examined, $(x, y) \in D$, reaches a steady state. Then by assumption, $\partial T/\partial t \to 0$, and the heat distribution is described by the equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{10.17}$$

This is Laplace's equation in 2D.

10.4 Separation of variables and onwards

Before we enter the next chapter where we study Fourier series, we introduce it here as a consequence of seeking certain solutions to the above partial differential equations. Consider first the 1D heat equation with fixed temperature conditions at the ends and a given initial temperature:

$$u_t = \kappa u_{xx}, \tag{10.18}$$

$$u(0,t) = 0 = u(L,t), \tag{10.19}$$

$$u(x,0) = f(x). \tag{10.20}$$

As a leap of faith, let us assume that the solution can be written in a separated form,

$$u(x,t) = X(x)G(t). \tag{10.21}$$

In this case, substitution into the PDE gives

$$G'X = \kappa X''G \Rightarrow \frac{G'(t)}{\kappa G(t)} = \frac{X''(x)}{X(x)}. \tag{10.22}$$

For convenience, we have moved the κ to the left hand-side, but this is not necessary. Now the key argument: in the above equation, the left hand-side is entirely a function of time while the right hand-side is a function of x. This means that both sides must be equal to a constant. *Why?* Consider if we were to change the value of x but keep t fixed. If the two sides were not constant, then necessarily the left hand-side, which only depends on time, must change its value in general.

Thus

$$\frac{G'(t)}{\kappa G(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 < 0. \tag{10.23}$$

The fact that we have taken the constant on the right hand-side to be a negative constant can only be seen *a posteriori* (that is, after we are done). Solving now the two equations for G and X yields[2]

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{10.24}$$

$$G(t) = D\mathrm{e}^{-\kappa\lambda^2 t}. \tag{10.25}$$

It remains to apply the two boundary conditions. Note that in order for G not to be trivial,

$$u(0,t) = 0 \Rightarrow X(0)G(t) = 0 \Rightarrow X(0) = 0 \tag{10.26}$$

$$u(L,t) = 0 \Rightarrow X(L)G(t) = 0 \Rightarrow X(L) = 0. \tag{10.27}$$

The first boundary condition tells us that $A = 0$ and the second tells us that

$$B\sin(\lambda L) = 0 \Rightarrow \lambda_n = \frac{\pi n}{L}, \quad n = 1, 2, 3, \ldots. \tag{10.28}$$

Notice that $n = 0$ is trivial, and moreover, negative integers, n simply lead to negating the arbitrary constant. Each value of λ_n leads to a possible solution. In general, we might consider adding together all the possible components in order to get a general form:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\kappa\lambda_n^2 t}\sin(\lambda_n x), \tag{10.29}$$

where $\lambda_n = \pi n/L$.

Now the last question we are left with is how to impose the initial condition, (10.20). If we substitute $t = 0$ into (15.11), we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n x}{L}\right). \tag{10.30}$$

This expression of a function into an infinity of sines (or cosines) is what is known as a Fourier series. The point here is that we wish for a way to solve for the unknown values B_n. At that both, we have a solution that formally satisfies the heat equation and all the necessary conditions.

[2]The differential equation for $G(t)$ is a first-order linear equation that can be solved through separation. The differential equation for $X(x)$ is a second-order constant-coefficient equation that can be solved by substitution of $X(x) = \mathrm{e}^{rx}$ and solving for r to find the two linearly independent solutions. Review your old notes on ODEs for a refresher.

Chapter 11

Fourier series I

At the end of the last chapter, we showed that by considering separable solutions to the heat equation, one naturally obtains solutions of the form (15.11):

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\kappa(\pi n/L)^2 t} \sin\left(\frac{\pi n x}{L}\right). \tag{11.1}$$

However, in order to solve for the unknown coefficients, B_n, we were required to impose the requirement of (10.30)

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n x}{L}\right), \tag{11.2}$$

where $f(x)$ is a given function. As we shall define in this chapter, the infinite sum to the right is known as Fourier (sine) series.

For simplicity, let us take $L = \pi$, and consider a sum where both cosine and sine terms are present. Thus consider an expression like,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]. \tag{11.3}$$

This is a Fourier series: a representation of a function, $f(x)$, through the addition of an infinite number of sines and cosines[1].

Take a moment to appreciate the radical nature of (11.3). Supposing that coefficients a_n and b_n could, in theory, be found and calculated. Then in a sense, what (11.3) indicates is that regardless of what $f(x)$ may ressemble, it can nevertheless be represented as a sum of cosines and sines. Thus for instance, $f(x) = x^2$ or $f(x) = e^x$, or any other function that does not ressemble sinusoidal functions whatsoever.

There are essentially two questions to investigating Fourier series such as (11.3). The first question is: given a particular f function, how does one compute the coefficients a_n and b_n? The second question is: is it true that the infinite sum can be equated to $f(x)$? The purpose of this chapter will be to answer the first question.

11.1 Periodic functions and even/odd functions

Definition 11.1 (Periodic)**.** A function f is periodic with period a if

$$f(x) = f(x+a) \qquad \forall\, x \in \mathbb{R}.$$

Note that the period is not unique, but if there is a smallest such a, it is called the prime or fundamental period of f.

[1] The writing of $a_0/2$ is just for convention. You will also notice that there is no need to add $b_0 \sin(0 \cdot x) = 0$

Example 11.2 (Periodic functions)**.** The constant function $f(x) = c$ is periodic but does not have a fundamental period. The functions $\sin(\pi x/L)$ and $\cos(\pi x/L)$ both have periods $2L$. Polynomial functions like x or x^2 are not periodic.

Note that if we have a function f defined on a half-open interval, say $(x_0, x_0 + a]$, we can extend it to be a periodic function by constructing shifted images of the function with period a (see Fig. 11.1). This is called a periodic extension.

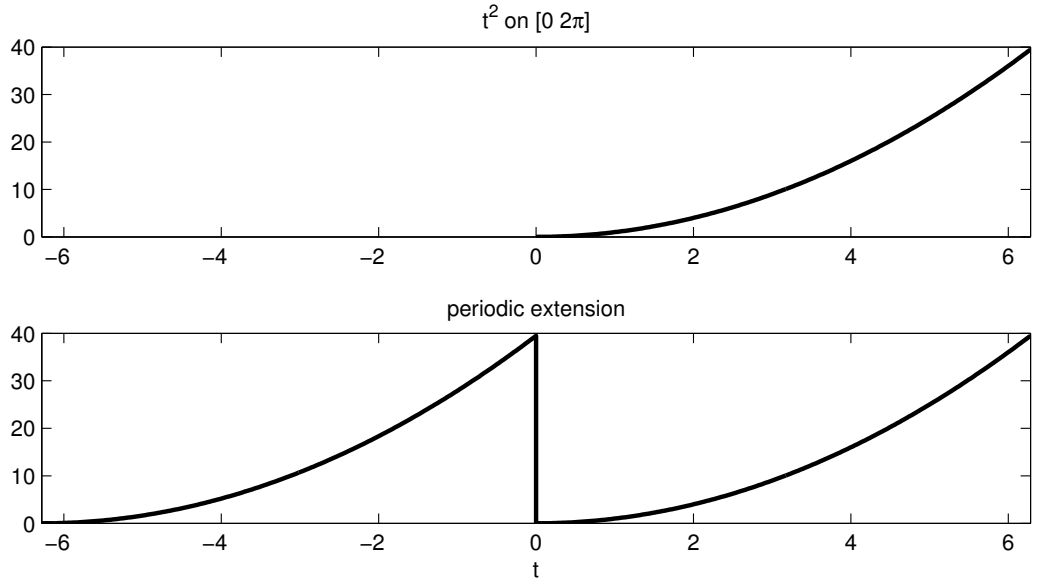


Figure 11.1: The periodic extension of $f(x) = x^2$. The function is originally defined on $(0, 2\pi]$ and then extended to the real line by creating copies of length 2π.

Definition 11.3 (Periodic extension to $\mathbb{R}$)**.** Given a function $f(x)$ defined for $x_0 < x \leq x_0 + a$, the *periodic extension*, F, is defined as follows. Given $x \in \mathbb{R}$, find the unique integer, m, such that $x - ma \in (x_0, x_0 + a]$. Then set

$$F(x) = f(x - ma).$$

Then F is a periodic function with period a.

Next we introduce the idea of even and odd functions.

Definition 11.4 (Odd function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is odd if

$$f(-x) = -f(x), \qquad \forall x \in \mathbb{R}. \tag{11.4}$$

Definition 11.5 (Even function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is even if

$$f(-x) = f(x), \qquad \forall x \in \mathbb{R}. \tag{11.5}$$

Lemma 11.6 (Properties of even and odd functions)**.**

1. For a **continuous** odd function, $f(0) = 0$.
2. For an odd function f,
$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0, \tag{11.6}$$
for any $a \in \mathbb{R}$.
3. For an even function g,
$$\int_{-a}^{a} g(x)\,\mathrm{d}x = 2\int_{0}^{a} g(x)\,\mathrm{d}x, \tag{11.7}$$
for any $a \in \mathbb{R}$.

4. The product of an odd and even function is odd. The product of two even functions is even. The product of two odd functions is even.

Proof. These all all proven readily using the definition of even and odd functions[2].

11.2 Fourier series for functions of period 2π

Let f be a function with period 2π. We shall seek to write f in the form of a Fourier series, with

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right), \tag{11.8}$$

where a_n and b_n are constants.

Let us make a few important remarks: first, note that since $\cos(nx)$ and $\sin(nx)$ are periodic with period 2π, so is the infinite sum. Second, it is not clear, at this point, whether or not the function $f(x)$ can be represented in the form (11.8). In particular, it is not clear that the infinite sum will converge. The equality here is to be interpreted as an *ansatz* or an assumed form. In particular, this inspires two key questions.

Remark 11.7 (Two questions regarding Fourier series)**.**

1. On the assumption that (11.8) holds, how can a_n and b_n be determined?
2. If a_n and b_n are determined, are there restrictions on whether or not the infinite series expansion is, in fact, equal to the original function $f(x)$?

Integrals of trigonometric functions

The use of Fourier series depends on several important consequences of integrating products of trigonometric functions. To begin, it is well worth noting that products of sines and cosines can be changed into sums as follows.

$$\sin(nx)\cos(mx) = \frac{1}{2}\Big[\sin((n-m)x) + \sin((n+m)x) \Big], \tag{11.9a}$$

$$\cos(nx)\cos(mx) = \frac{1}{2}\Big[\cos((n-m)x) + \cos((n+m)x) \Big], \tag{11.9b}$$

$$\sin(nx)\sin(mx) = \frac{1}{2}\Big[\cos((n-m)x) - \cos((n+m)x) \Big]. \tag{11.9c}$$

Lemma 11.8 (Orthogonality Lemma)**.** If $n, m \in \mathbb{Z}$, with $n, m \in \mathbb{Z} \setminus 0$, then we have the following equalities:

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\ \mathrm{d}x = 0, \tag{11.10a}$$

$$\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ \mathrm{d}x = \pi\delta_{mn}, \tag{11.10b}$$

$$\int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ \mathrm{d}x = \pi\delta_{mn}, \tag{11.10c}$$

where recall from the definition of the Kronecker (A.1),

$$\delta_{mn} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \tag{11.11}$$

The cases where both $m = n = 0$ should be done separately.

[2]Note that although Definition 11.4 applies to $\mathbb{R}$, it is possible to consider an odd function discontinuous at $x = 0$, and where (11.6) is true.

Proof.

1. If $n \neq m$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\ \cos(mx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}\Big[\sin((n-m)x) + \sin((n+m)x)\Big]\,\mathrm{d}x,\\ &= \frac{1}{2}\left[-\frac{\cos((n-m)x)}{n-m} - \frac{\cos((n+m)x)}{n+m}\right]_{-\pi}^{\pi},\\ &= 0.\end{aligned}$$

   If $n = m$ the proof is similar, with the first term disappearing since $\sin(0 \cdot x) = 0$.

2. If $n \neq m$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}\Big[\cos((n-m)x) + \cos((n+m)x)\Big]\,\mathrm{d}x,\\ &= \frac{1}{2}\left[\frac{\sin((n-m)x)}{n-m} + \frac{\sin((n+m)x)}{n+m}\right]_{-\pi}^{\pi},\\ &= 0.\end{aligned}$$

   If $n = m \neq 0$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}\Big[\cos(0x) + \cos(2mx)\Big]\,\mathrm{d}x,\\ &= \frac{1}{2}\left[x + \frac{\sin(2mx)}{2m}\right]_{-\pi}^{\pi},\\ &= \pi.\end{aligned}$$

   If $n = m = 0$

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\ \cos(mx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} 1 \cdot 1\,\mathrm{d}x,\\ &= 2\pi.\end{aligned}$$

3) The case $m = n = 0$ is straightforward.

   If $m \neq n$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\ \sin(mx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}\Big[-\cos((n+m)x) + \cos((n-m)x)\Big]\,\mathrm{d}x,\\ &= \frac{1}{2}\left[-\frac{\sin((n+m)x)}{n+m} + \frac{\sin((n-m)x)}{n-m}\right]_{-\pi}^{\pi},\\ &= 0.\end{aligned}$$

   If $m = n$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ \mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}\Big[-\cos((2m)x) + \cos((0)x)\Big]\,\mathrm{d}x,\\ &= \frac{1}{2}\left[-\frac{\sin(2mx)}{2m} + x\right]_{-\pi}^{\pi},\\ &= \pi.\end{aligned}$$

As an interesting aside: the orthogonality lemma shows that the functions $1, \sin x, \cos x, \sin 2x, \ldots$ form an orthogonal basis for the infinite-dimensional vector space consisting of 2π-periodic functions whose Fourier series converge.

Fourier coefficients

We are now ready to address the first question of Remark 11.7, that is, on the assumption that $f(x)$ is written as (11.8), what are the values of a_n and b_n?

First, we multiply (11.8) by $\cos(nx)$ and integrate. This gives

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\ \cos(mx) + b_m\ \sin(mx)\right)\right)\cos(nx)\ \mathrm{d}x.$$

Assuming we can interchange the integral and the sum, we then have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \int_{-\pi}^{\pi}\frac{a_0}{2}\ \cos(nx)\ \mathrm{d}x \\ + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos(mx)\cos(nx)\ \mathrm{d}x \\ + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi}\sin(mx)\cos(nx)\ \mathrm{d}x. \quad (11.12)\end{aligned}$$

Using (11.10a), the last term vanishes (since each of the integral of the third sum is zero) and thus

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x = \int_{-\pi}^{\pi}\frac{a_0}{2}\ \cos(nx)\ \mathrm{d}x \\ + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos(mx)\cos(nx)\ \mathrm{d}x. \quad (11.13)\end{aligned}$$

This leads to two cases, dependent on whether $n \neq 0$ or $n = 0$.

Case 1: $n \neq 0$. We have

$$\int_{-\pi}^{\pi}\frac{a_0}{2}\ \cos(nx)\ \mathrm{d}x = 0,$$

and

$$\sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos(mx)\ \cos(nx)\ \mathrm{d}x = \pi a_n$$

by (11.10b) of the orthogonality lemma (if $m = n$ the integral equals π, otherwise it equals zero). Therefore (11.13) becomes

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ \mathrm{d}x.$$

Case 2: $n = 0$. It is important to explicitly check the edge cases. Returning to (11.13), we have

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi}\frac{a_0}{2}\,\mathrm{d}x + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos(mx)\ \mathrm{d}x.$$

The last term is zero by (11.10b) of the orthogonality lemma, and thus (11.13) becomes

$$a_0 = \frac{1}{\pi}\ \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x. \quad (11.14)$$

This confirms that the formula (11.2) applies with $n = 0$.

In order to develop the equations for b_n, we again return to (11.8), multiply both sides by $\sin(mx)$ and integrate. This gives,

$$b_n = \frac{1}{\pi}\ \int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x. \quad (11.15)$$

The above proves the following main result of this chapter.

Theorem 11.9 (Fourier coefficients of 2π periodic f)**.** Let f be a 2π periodic function. Suppose that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x, \tag{11.16a}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x \tag{11.16b}$$

exist. Then we shall write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right), \tag{11.16c}$$

and call the series on the right hand-side the *Fourier series* for f, whether or not it converges to f. The coefficients a_n and b_n are called the Fourier coefficients of f.

Remark 11.10 (Tilde notation of Fourier series)**. Caution!**

The astute reader will notice that we have changed from using equality, like in (11.8) to the tilde in (11.16c). This is ambiguity associated with the manipulation of infinite series where the convergence is unclear[3]. In essence, when we write (11.8), we are positing a possible form for $f(x)$, without any regard to the fact that the form, itself, may not be permissible.

The use of the tilde then specifies to great precision that the representation may not be true. In lectures, to emphasize the use of the tilde, we will say "is similar to" or "is written as". The same is true caveats are true in writing the geometric series as,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \tag{11.17}$$

where we are accustomed to considering equality first, and then questioning convergence second. Here, equality is only sensible if $|x| < 1$, as the series diverges if $|x| \geq 1$.

The key is that in the next chapter, we clarify to what extent the tilde or similarity, $\sim$, is like $=$. In general, it is not disastrous to mix the notation because it is generally clear from the context how to interpret the expression.

11.3 Fourier sine and cosine series

Lemma 11.11 (Fourier coefficients of even and odd functions)**.**
If $f(x)$ is even, then $b_n = 0$ for all n, and thus

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \\ \text{where } a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, \mathrm{d}x, \quad \text{for } n \geq 0. \end{aligned} \tag{11.18}$$

If $f(x)$ is odd then $a_n = 0$ for all n, and thus

$$\begin{aligned} f(x) &\sim \sum_{n=1}^{\infty} b_n \sin(nx), \\ \text{where } b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x, , \quad \text{for } n \geq 1. \end{aligned} \tag{11.19}$$

[3]The same convention is used in e.g. p. 74 of [**?**]

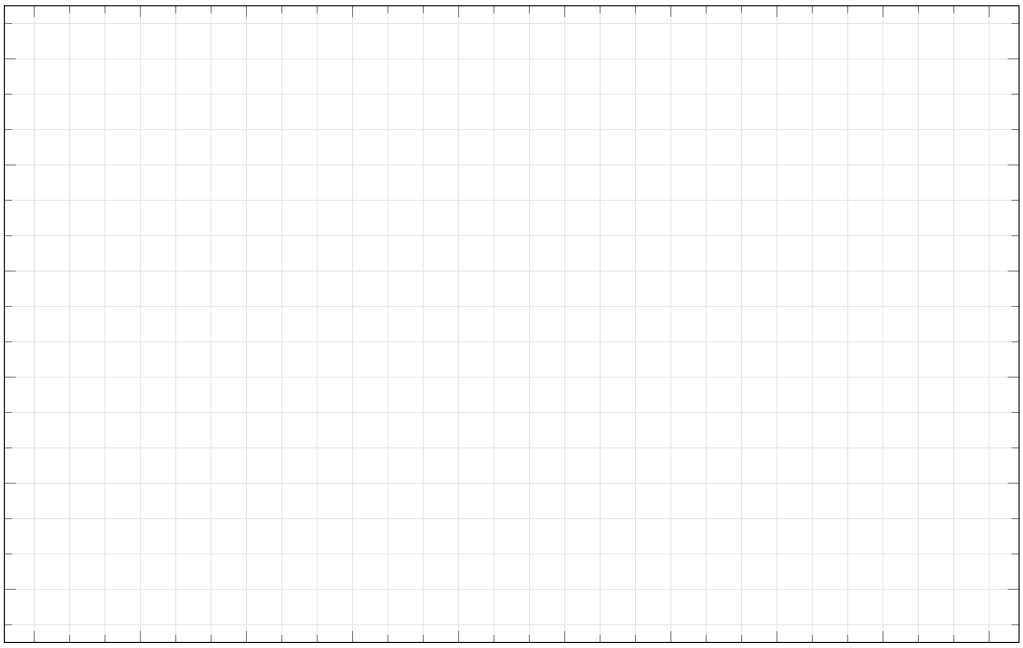
Proof. This is readily proven by the even and odd properties of Lemma 11.6. For instance, let us assume that $f(x)$ is even. Then

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = 0,$$

since $f(x)\sin(nx)$ is an odd function integrated about the origin. A very similar argument shows that $a_n = 0$ when f is odd.

Example 11.12. Find the Fourier series of the function f which is periodic with period 2π such that

$$f(x) = |x|, \quad -\pi < x \leq \pi. \tag{11.20}$$



Solution. Firstly, note that f is an even function. Hence $f(x)\sin(nx)$ is odd and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = 0. \tag{11.21}$$

Alternatively, this is a confirmation of Lemma 11.11. Calculating now the Fourier cosine coefficients,

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x. \end{aligned} \tag{11.22}$$

This result is integrated by parts to give

$$\begin{aligned} a_n &= \frac{2}{\pi}\left\{\left[\frac{x\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,\mathrm{d}x\right\} \\ &= -\frac{2}{\pi}\left[\frac{-\cos(nx)}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{\cos(n\pi)-\cos(0)}{n^2}\right] \\ &= \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}, \end{aligned}$$

where the last line follows from $\cos(n\pi) = (-1)^n$.

When performing computations as above, it is crucial to inspect the operations and note any zeros in the denominator. In the case of the above, the calculation does not apply if $n = 0$. One can return to (11.22) and substitute $n = 0$. This gives

$$a_0 = \frac{2}{\pi} \int_0^\pi x \, \mathrm{d}x = \pi.$$

The Fourier series is thus

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{\pi} \frac{(-1)^n - 1}{n^2} \right] \cos(nx).$$

However, it is customary to simplify the expression further. Note that $a_n = 0$ when n is even, so the above sum will contain a zero every second term. Thus we set $n = 2m + 1$. This gives,

$$a_{2m+1} = \frac{2(-1-1)}{\pi(2m+1)^2} = -\frac{4}{\pi(2m+1)^2}, \qquad m \geq 0$$

Thus the Fourier series is more appropriately written as

$$f(x) \sim \frac{\pi}{2} + \sum_{m=0}^{\infty} \left[-\frac{4}{\pi(2m+1)^2} \right] \cos[(2m+1)x].$$

Chapter 12

Fourier series II

In the previous chapter, we assumed that a 2π-periodic function could represented in the form of a Fourier series, and we wrote

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right). \tag{12.1}$$

We subsequently derived formulae for the coefficients a_n and b_n. This dealt with Question 1 of Remark 11.7. However, the second question has not been answered. In effect, the second question asks whether we can establish the convergence of the infinite series on the right to the function $f(x)$ (and under what conditions). In other words, to what extent is $\sim$ like an equality?

Establishing convergence is far from an elementary problem. Your standard calculus tests (like the ratio and root tests) are inconclusive. Even beyond the criteria of convergence, establishing to what function the infinite series converges to is a different matter as well. Recall that the convergence of an infinite series depends on the convergence of its sequence of partial sums, which are given by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right). \tag{12.2}$$

Thus a Fourier series converges to a point x if and only if its partial sums have a limit:

$$\lim_{N\to\infty} S_N(x) = \tilde{f}(x), \tag{12.3}$$

which may or may not equal the value of the original function $f(x)$.

This raises two important questions:

Q1 Does the limit of (12.3) exist? And for what x?

Q2 Is $\sim$ in fact equivalent to $=$ in (12.1).

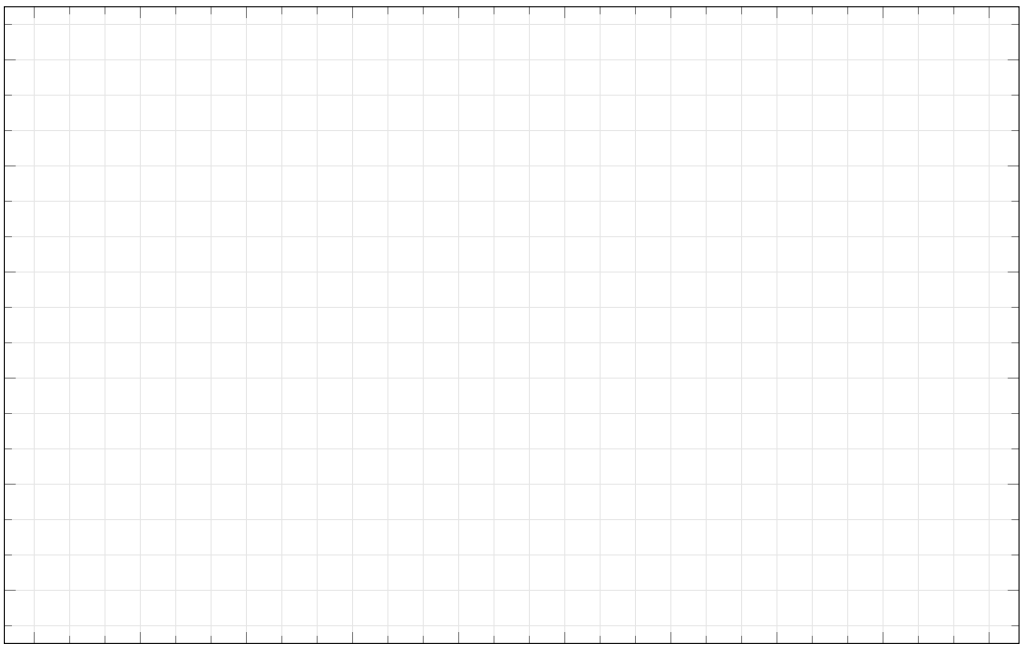
We will give partial answers to both questions. Firstly, however, we go through a classic example to demonstrate that convergence is perhaps not as straightforward as might be believed.

Example 12.1 (Square Wave)**.** Find the Fourier series of the following function:

$$f(x) = \begin{cases} -1 & -\pi < x < 0, \\ 1 & 0 < x < \pi, \\ 0 & t = 0,\, \pi, \end{cases}$$

and

$$f\left(x + 2\pi\right) = f\left(x\right).$$



Solution. Using Definition 12.7 with $L = \pi$ we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0,$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos(nx)\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi}(1)\cos(nx)\,\mathrm{d}x \\ &= -\frac{1}{\pi n}\big[\sin(nx)\big]_{-\pi}^{0} + \frac{1}{\pi n}\big[\sin(nx)\big]_{0}^{\pi} = 0. \end{aligned}$$

Also,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(nx)\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin(nx)\,\mathrm{d}x \\ &= \frac{1}{\pi n}\big[\cos(nx)\big]_{-\pi}^{0} - \frac{1}{\pi n}\big[\cos(nx)\big]_{0}^{\pi}. \end{aligned}$$

Remembering that $\cos n\pi = (-1)^n$, we see that

$$\begin{aligned} b_n &= \frac{1}{\pi n}\Big(1-(-1)^n\Big) - \frac{1}{\pi n}\Big((-1)^n - 1\Big), \\ &= \frac{2}{\pi n}\Big(1-(-1)^n\Big), \\ &= \begin{cases} \dfrac{4}{n\pi} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \end{aligned} \tag{12.4}$$

$$f(x) = \lim_{N\to\infty} S_N(x) = \lim_{N\to\infty}\left[\frac{4}{\pi}\sum_{n=1,\,n\text{ odd}}^{N}\frac{\sin(nx)}{n}\right].$$

where we have shifted the above index $n \mapsto 2k+1$.

Before moving on, note that in the case of Fourier series where the non-trivial coefficients proceed in even or odd indexes, it is common to shift the index of summation so as to increase in unit steps. Thus from (12.4), we set $n = 2m+1$ for $m = 0, 1, 2, \ldots$, so that

$$b_{2m} = \frac{4}{(2m+1)\pi}. \tag{12.5}$$

The Fourier series is now written as

$$f(x) = \lim_{N\to\infty} S_N = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin[(2m+1)x}{(2m+1)}. \tag{12.6}$$

Gibbs Phenomenon

We want to now return and examine the effect of adding term-by-term in forming the infinite series. The first few partial sums yield,

$$S_1(x) = \frac{4}{\pi}\sin x, \quad S_3(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right).$$

Figure 12.1 shows S_N with $N = 1, 3, 5, 7, 9$ and 99, all plotted on the same axes. Note the oscillations at multiples of π (i.e. where the function jumps); the wavelength of these oscillations decreases as $N \to \infty$, but the amplitude stays constant. The presence of these oscillations is called the *Gibbs Phenomenon*[1].

Looking back at the first important question post at the beginning of this chapter, we see that in this example, two aspects can be remarked. Firstly, it looks like the sequence of partial sums, S_N approaches f pointwise, that is, for every x, $S_N(x) \to f(x)$ as $N \to \infty$. Secondly, it looks like the error between the Nth approximation and the exact solution,

$$\|S_N - f\|_\infty = \max_x |S_N(x) - f(x)|, \tag{12.7}$$

does not tend to zero as $N \to \infty$, due to Gibbs phenomenon.

12.1 Fourier convergence

Based on the example of the square wave, we have observed that the Fourier series seems to converge in a pointwise manner upon the exact function (the square wave), except at points of discontinuity, $x = n\pi$. At such points, the Fourier series seems to converge to zero. To make this more precise, let us introduce the notion of a one-sided limit.

Definition 12.2 (One-sided limits)**.** In the limit that x approaches c from the right, we write, for the limiting value of $f(x)$,

$$f(c_+) = \lim_{\substack{h\to 0\\ h>0}} f(c+h),$$

if it exists. We then call this the right-hand limit of f at c. Similarly, the left-hand limit of f at c is defined to be

$$f(c_-) = \lim_{\substack{h\to 0\\ h>0}} f(c-h),$$

if it exists.

[1]For every N, we can find a point t_N corresponding to a peak of one of the oscillations near the point π (i.e. a point where f jumps). Since the amplitude of the oscillations does not tend to zero as $N \to \infty$, this shows S_N can't converge uniformly to f (note that the points t_N tend to π as $N \to \infty$).

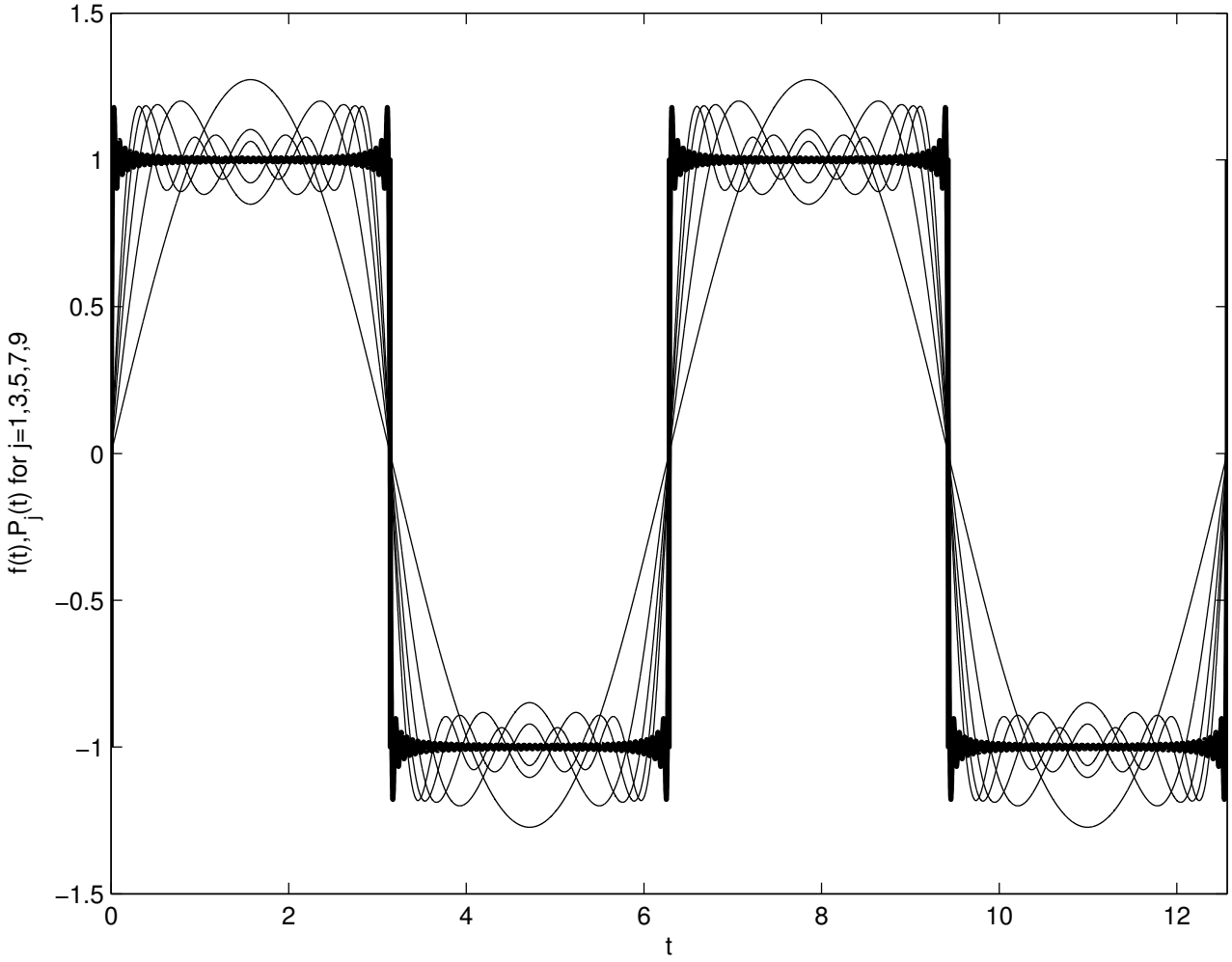


Figure 12.1: Figure showing the square wave, S_N with $N = 1, 3, 5, 7, 9$ and 99, all plotted on the same axes.

Example 12.3. For the square wave of eqn (12.1), $f(0-) = -1$ and $f(0^+) = 1$. Also, functions do not always have to have left and right-handed limits. For instance, $\sin(1/x)$ has neither as $x \to 0$.

Definition 12.4 (Piecewise continuous)**.** The function f is piecewise continuous on (a, b) if we can divide (a, b) into a finite number of sub-intervals, where: (i) f is defined and continuous on each; and (ii) the left- and right-hand limits at the endpoints of each sub-interval exist.

Theorem 12.5 (Fourier convergence theorem)**.** Let f be a 2π-periodic function, with f and f' piecewise continuous on $(-\pi, \pi)$. Then the Fourier series of f at x converges to the average of the left- and right-limits. That is,

$$\frac{1}{2}[f(x_-) + f(x_+)] = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]. \tag{12.8}$$

Example 12.6. In the square-wave function (12.1) with Fourier series (12.6), we have by Theorem 12.5 that at any point, x,

$$\frac{1}{2}[f(x_-) + f(x_+)] = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)x]}{(2m+1)}. \tag{12.9}$$

Notice that at $x = 0, \pm\pi$, both sides reduce to zero. Similarly at $x = \pi/2$, we have

$$1 = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)\frac{\pi}{2}]}{(2m+1)} = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)}, \tag{12.10}$$

and thus we are able to conclude that

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)}. \tag{12.11}$$

12.2 Functions of any period

Theorem 11.9 shows that the Fourier coefficients of 2π periodic function are given by (11.16a). For a general periodic funciton f with period $2L$, the formula (11.16a) can be adapted to yield the Fourier coefficients.

Theorem 12.7 (Fourier coefficients for a $2L$-periodic function)**.** Let f be a periodic function with period $2L$. Then the Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \tag{12.12a}$$

where the *Fourier coefficients* of f are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \tag{12.12b}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{12.12c}$$

Proof. We perform a coordinate stretching of the x domain so that the function is 2π periodic. For example, this should map $\cos(n\pi x/L)$ to $\cos(nX)$. Thus, set

$$X = \frac{\pi x}{L},$$

and define

$$g(X) = f(x) = f(LX/\pi).$$

It can be verified firstly that $g(X)$ is 2π periodic in X, since

$$g(X + 2\pi) = f\left(\frac{L(X+2\pi)}{\pi}\right) = f(LX/\pi + 2L) = f(LX/\pi) = g(X),$$

based on the $2L$-periodicity of f. Hence we can write the Fourier series for $g(X)$:

$$g(X) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nX) + b_n \sin(nX) \right],$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(X) \cos(nX) \, \mathrm{d}X \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(X) \sin(nX) \, \mathrm{d}X.$$

Now take the above formulae and re-substitute $x = LX/\pi$. Simplification results in the stated formulae.

Remark 12.8. Note that in Theorem 12.7, we have assumed that the Fourier series of the $2L$-periodic function is integrated via $[-L, L]$ but this is not necessary. Consider for instance a function originally defined on $[0, 2L]$ and then periodically extended to $\mathbb{R}$. In this case, it may be more convenient to integrate instead,

$$a_n = \frac{1}{L} \int_{0}^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{0}^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

which is equivalent to (12.12) on account of the periodicity.

12.3 Periodic Extensions

In Theorem 12.7 above, we demonstrated how the Fourier series of any $2L$-periodic function could be derived. In some applications, we wish to consider a function f defined on $[0, L]$ and construct a Fourier series with only cosine terms or sine terms. This is done by extending f in an even manner (for cosine terms) to $[-L, L]$, and then extending this to the usual $2L$-periodic function. The result is cosine series defined on the original $[0, L]$ domain. The same principal can be used to construct sine series.

Definition 12.9 (Even and odd periodic extensions)**.** Given $f(x)$ defined on $x \in [0, \pi]$, its *even periodic extension* is defined by

$$f_e(x) = \begin{cases} f(x) & x \in [0, L], \\ f(-x) & x \in [-L, 0). \end{cases}$$

so that we have $f_e(-x) = f_e(x)$ for $x \in [-L, L]$. Similarly its *odd periodic extension* is defined by

$$f_o(x) = \begin{cases} f(x) & x \in [0, L], \\ -f(-x) & x \in [-L, 0). \end{cases}$$

so that we have $f_o(-x) = -f_o(x)$. Once both functions have been extended to $[-L, L]$, they can be further extended to a $2L$-periodic function over all x in the usual way. So for instance,

$$f_e(x + 2kL) = f_e(x) \quad \text{for } k \in \mathbb{Z},\ x \in (-L, L). \tag{12.13}$$

Example 12.10. Fig. 12.2 shows the the odd/even periodic extensions of $f(x) = x^2$.

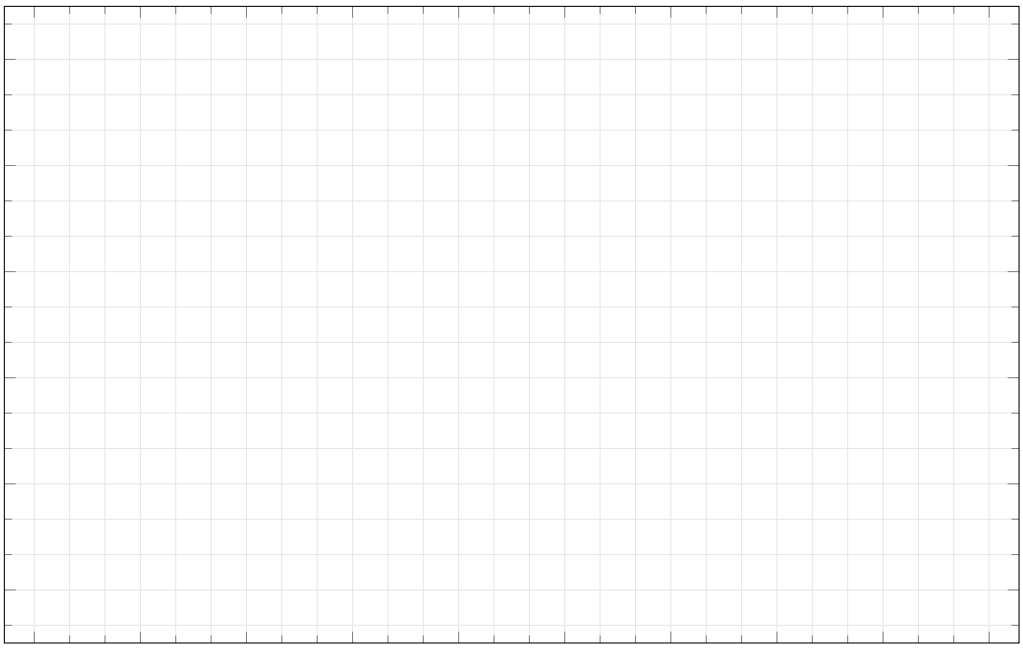


Figure 12.2: The odd and even periodic extensions of $f(x) = x^2$.

Recall from Lemma 11.11 that for an even function defined on $[-L, L]$, the Fourier coefficients $b_n = 0$ for all n. Thus for an even extension,

$$\begin{gathered} f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \\ \text{where } a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \qquad n \geq 0. \end{gathered} \tag{12.14}$$

Similarly for an odd extension,

$$\begin{aligned} f_o(x) &\sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \\ \text{where } b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad n \geq 1. \end{aligned} \tag{12.15}$$

Example 12.11. Construct the Fourier series of the periodic extension of e^x on $(0, 2\pi)$.

Solution. Using Theorem 12.7 and Remark 12.8 with $L = \pi$, we have that

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \mathrm{e}^x \cos(nx)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \mathrm{e}^x \sin(nx)\,\mathrm{d}x.$$

Recall that

$$\begin{aligned} \int \mathrm{e}^{\alpha x}\cos(\beta x)\,\mathrm{d}x &= \frac{1}{\alpha^2+\beta^2}\mathrm{e}^{\alpha x}\Big[\alpha\cos(\beta x) + \beta\sin(\beta x)\Big] + c, \\ \int \mathrm{e}^{\alpha x}\sin(\beta x)\,\mathrm{d}x &= \frac{1}{\alpha^2+\beta^2}\mathrm{e}^{\alpha x}\Big[\alpha\sin(\beta x) - \beta\cos(\beta x)\Big] + c, \end{aligned} \tag{12.16}$$

and using these formulae we find that

$$\begin{aligned} a_n &= \frac{1}{\pi(1+n^2)}\left(\mathrm{e}^{2\pi}-1\right), \\ b_n &= -\frac{n}{\pi(1+n^2)}\left(\mathrm{e}^{2\pi}-1\right). \end{aligned} \tag{12.17}$$

Figure 12.3 shows the partial sums of the Fourier series, S_N for $N = 1, \ldots, 20$.

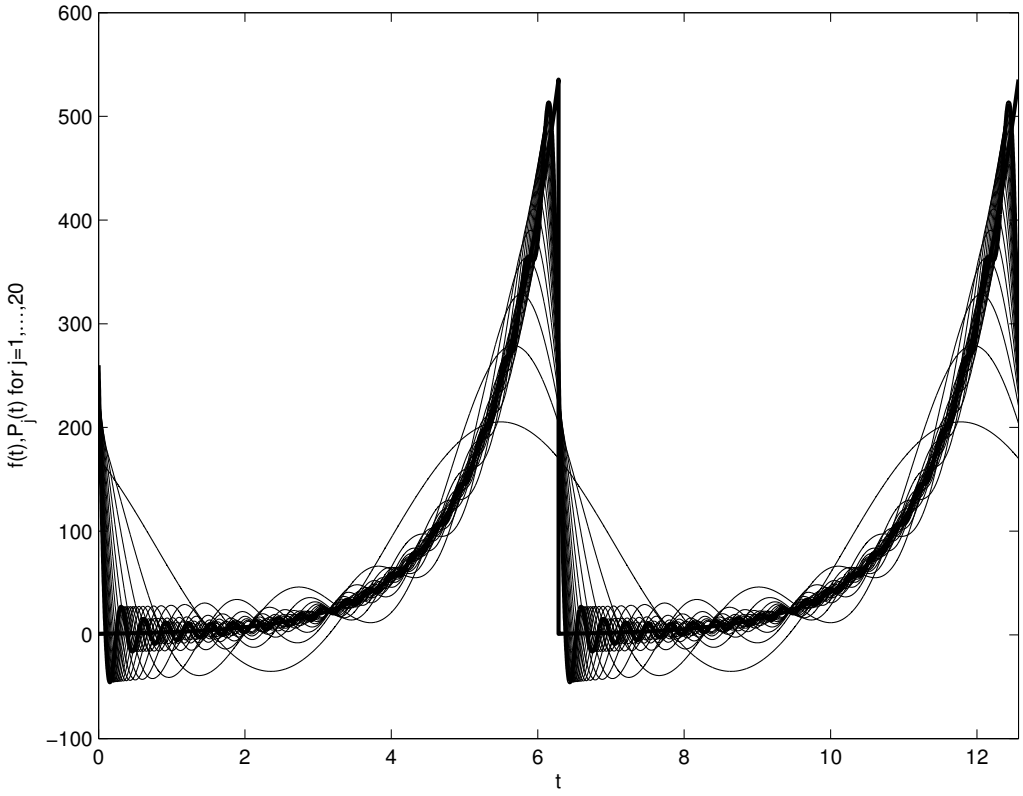


Figure 12.3: S_N for $N = 1, \ldots, 20$ for Fourier series of the periodically extension of $\exp(t)$.

Example 12.12. Construct the Fourier cosine series of the even extension of e^x on $(0, \pi)$.

Thus for this question, we consider the even extension,

$$f_e(x) = \begin{cases} \mathrm{e}^{x} & x \in [0, \pi], \\ \mathrm{e}^{-x} & x \in [-\pi, 0], \end{cases} \tag{12.18}$$

which then forms a function on $[-\pi, \pi]$ and is then periodically extended to the real line. Thus $L = \pi$, and from (12.14),

$$a_n = \frac{2}{\pi} \int_0^{\pi} \mathrm{e}^{x} \cos(nx)\, \mathrm{d}x = \frac{2}{\pi(1+n^2)} \Big((-1)^n \mathrm{e}^{\pi} - 1 \Big). \tag{12.19}$$

Figure 12.4 shows the partial sums, S_N for $N = 1, \ldots, 20$.

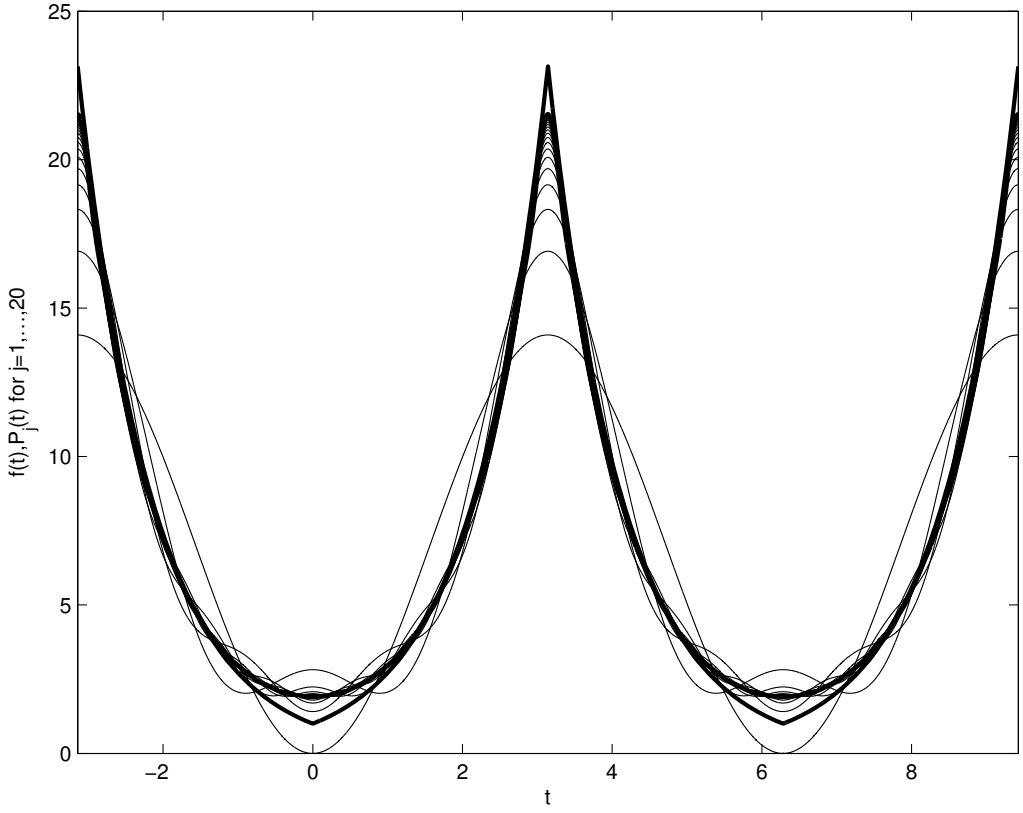


Figure 12.4: S_N for $N = 1, \ldots, 20$ for the cosine series of the even periodic extension of $\exp(t)$.

Comparing Figures 12.3 and 12.4 (and noting the different scales on the axes) we see that the cosine series of the even periodic extension converges faster than the full Fourier series of the regular periodic extension. Moreover, we may observe that

- the periodic extension of $\exp(x)$ from $(0, 2\pi)$ to $\mathbb{R}$ is *discontinuous* (as a function on $\mathbb{R}$), but the even periodic extension of $\exp(x)$ from $(0, \pi)$ to $\mathbb{R}$ is *continuous*;
- the coefficients in the Fourier series of $\exp(x)$, (12.17), decay slower than the coefficients in the Fourier cosine series of the even extension of $\exp(x)$, (12.19), ($1/n$ versus $1/n^2$).

This example illustrates the important fact that the smoother the function, the faster the decay of its Fourier coefficients, and hence the faster the convergence of its Fourier series. In general, the more derivatives f possesses, the faster the convergence. In fact, it can be roughly said that if the discontinuity is in the pth derivative, then the coefficients a_n and b_n decay as $1/n^{p+1}$.

Chapter 13

Maths of music

This is a short supplement to complement an in-class discussion on how the theory of Fourier series or Fourier analysis can be used for analysing audio.

If you bang a drum, this causes the surface of the drum to vibrate. That vibration is then transfered to vibrations in the air molecules. The disturbance propagates through the air, and vibrates your ear drums, which is then interpreted as sound.

As we know, sound can be recorded using a microphone, and the process is as follows. A microphone consists of a diaphragm—a thin piece of material that vibrates. The vibration causes a magnetic component to oscillate, and this oscillation induces a current in a circuit, which is recorded. Thus a sound file is essentially a record of these vibrations (as a function of time).

13.1 Make a sound

Let f be the frequency of pure sinusoidal note, as measured in hertz, Hz[1]. The note is given by

$$f(t) = \sin(2\pi f t) \quad \text{for } 0 \leq t \leq t_f, \tag{13.1}$$

and the amplitude is normalized to one. The digital representation of this note will not be continuous in time, but will instead record the note at a certain sampling frequency, F_s. In other words, we evaluate the above function from $t = 0$ to $t = t_f$ in steps of $\Delta t = 1/F_s$. This is done with the following code.

```
Fs = 2^(13); T = 5;
t = 0:(1/Fs):T;
f = sin(2*pi*440*t);
sound(f, Fs);
```

The above plays a 440 Hz note for $t_f = 5$ s with a sampling frequency of $F_s = 2^{13}$ Hz. This corresponds to the A string on a violin. If we like, we can bring in the E string as well,

```
f = 0.2*sin(2*pi*440*t) + sin(2*pi*659.3*t);
```

The two notes are shown in Fig. 13.1.

13.2 Analyze a sound

We have to introduce a slightly different notation so that it lines up with conventional signal processing. In our above numerical representation, the note is given by a vector of N values of f_n through $n = 0, \ldots, N-1$. It is perhaps better to visualize the vector in terms of the stem plot of Fig. 13.2. This shows the first 100 values.

[1] One hertz is equal to one cycle per second. It has SI units of s^{-1}.

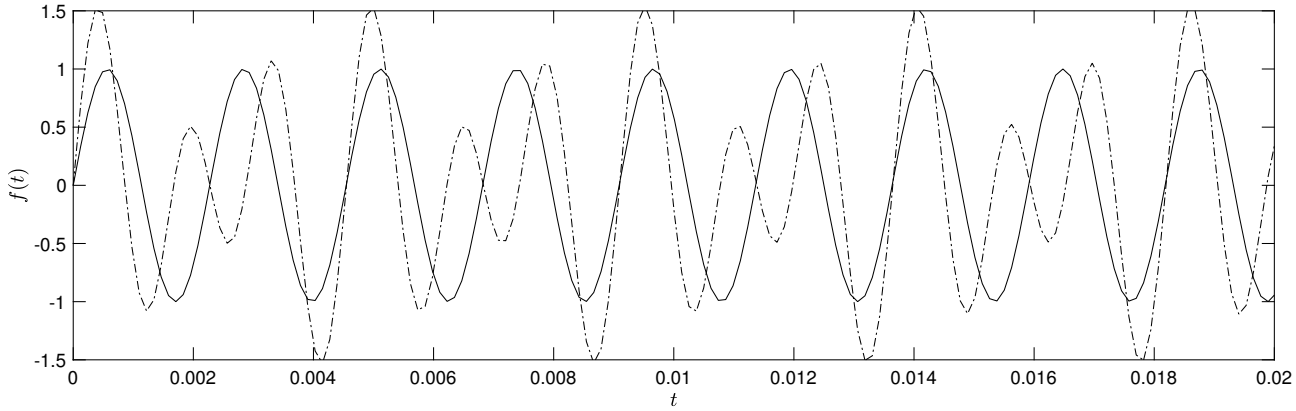


Figure 13.1: (solid) The A note at 440 Hz and (dash-dotted) the combined AE note at 440 Hz and 659.3 Hz.

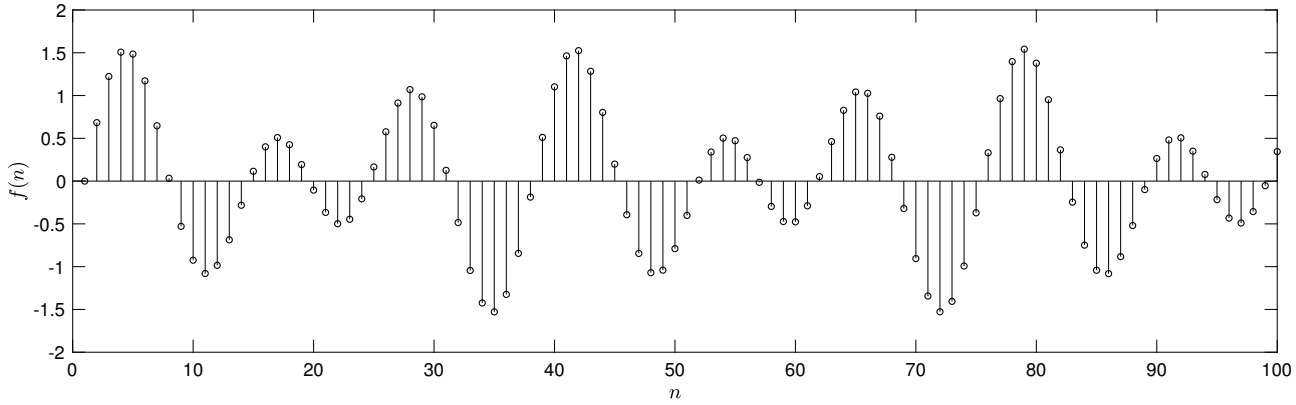


Figure 13.2: The combined AE note at 440 Hz and 659.3 Hz plotted in a stem plot to show the discrete nature.

Let us now write

$$f(n) = \sum_{k=0}^{N-1} F_k \left[\cos\left(\frac{k\pi n}{(N/2)} \right) - i \sin\left(\frac{k\pi n}{(N/2)} \right) \right], \tag{13.2}$$

where $F_k \in \mathbb{C}$. This looks awfully familiar. Compare it to (12.12). If F_k is a complex value, now, and we think of the domain as $n \in [0, N]$ then this looks like the Fourier series of an N-periodic function. It is[2]. We prefer to write

$$f(n) = \sum_{k=0}^{N-1} F_k \exp\left[i \left(\frac{2\pi k}{N} n \right) \right]. \tag{13.3}$$

The Fourier coefficients are now given by the discrete analog to the expressions you have learned. They are,

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f(n) \exp\left(-i \frac{2\pi k}{N} n \right). \tag{13.4}$$

The expression above is known as the Discrete Fourier Transform (DFT). In essence, given any vector of length N with values $f(n)$, it allows you to calculate the coefficients F_k.

In Matlab, the F_k values can be calculated using the command `fft`. In Fig. 13.3, we show the spectrum of the AE note. The amplitude $|F_k|$ gives the measure of how large the kth cosine or sine oscillation will be. As expected, if we scale the horizontal axis correctly to return the Hz units, then we see a spike at 440 Hz and one at 659.3 Hz.

[2] see for instance §5.7.2 of the book [**?**]

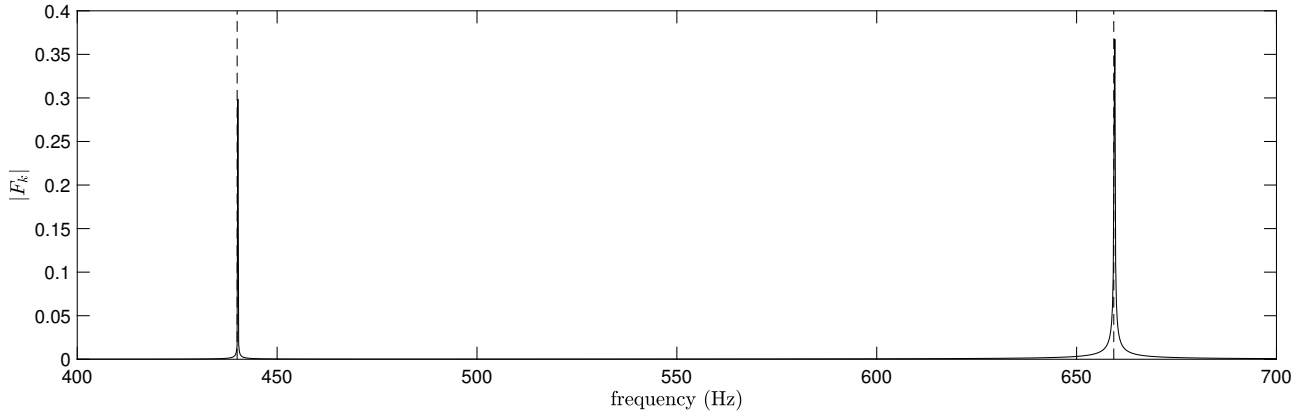


Figure 13.3: Spectrum of the AE note.

13.3 Real notes and sounds

We will play the A and AE note in lectures, and you will probably remark that this doesn't sound very much like was the same note sounds on a real instrument. The waveform for a real violin is shown in Fig. 13.4. Indeed, we see that it is much more complicated than the pure sinusoidal we had generated earlier.

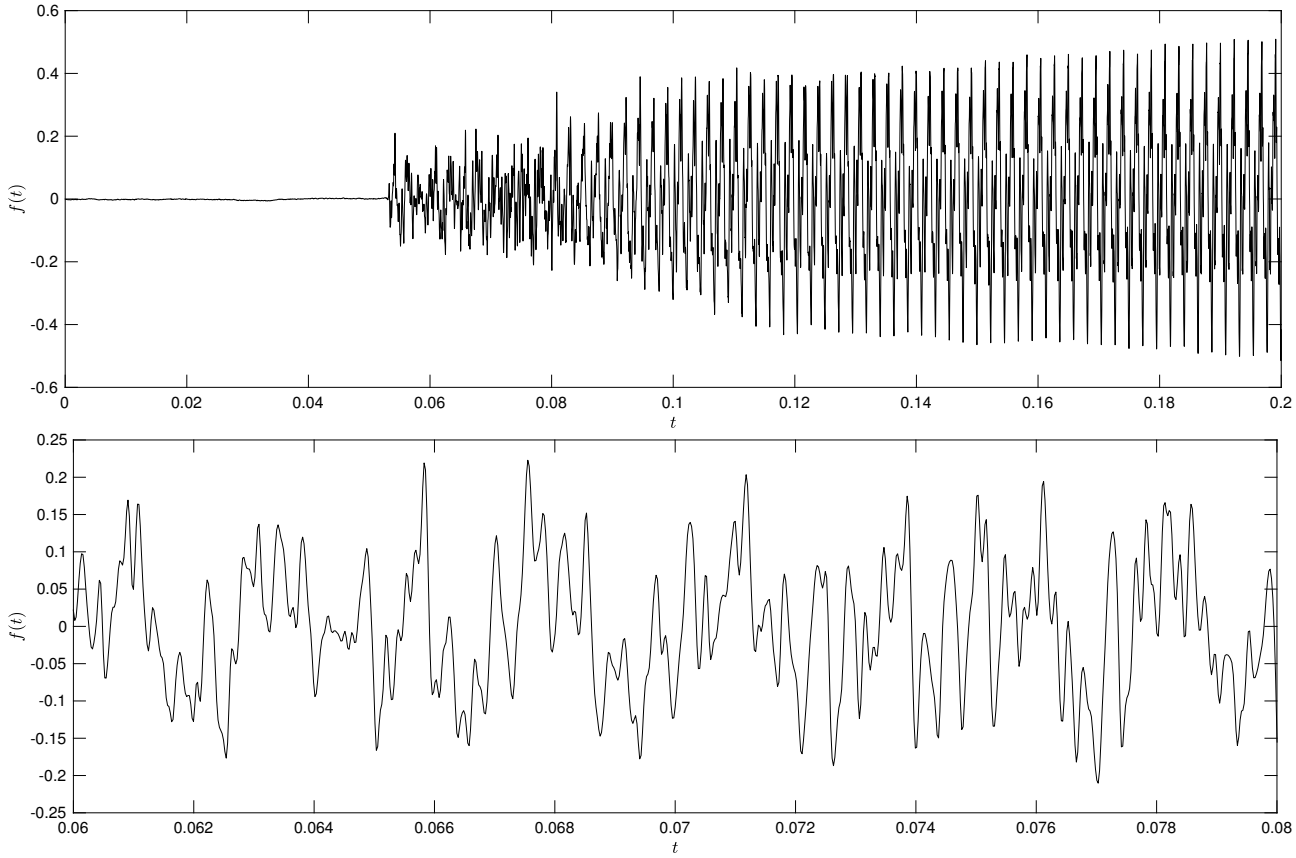


Figure 13.4: Waveform of a real A note played on a violin, shown for f vs. t. The top plot shows time up to 0.2s while the bottom shows a blowup between 0.06s and 0.08s.

The spectrum is shown in Fig. 13.5.

What we see is that we no longer get a pure 440 Hz tone. While the dominant peak happens at 440 Hz, we also get peaks at multiples of this frequency, hence 880 Hz and 1320 Hz. This is called acoustic resonance, and is what gives instruments their richness in contrast to a pure synthetic sound.

One final note. The above spectrum only gives a measure of the spectrum through the entire period

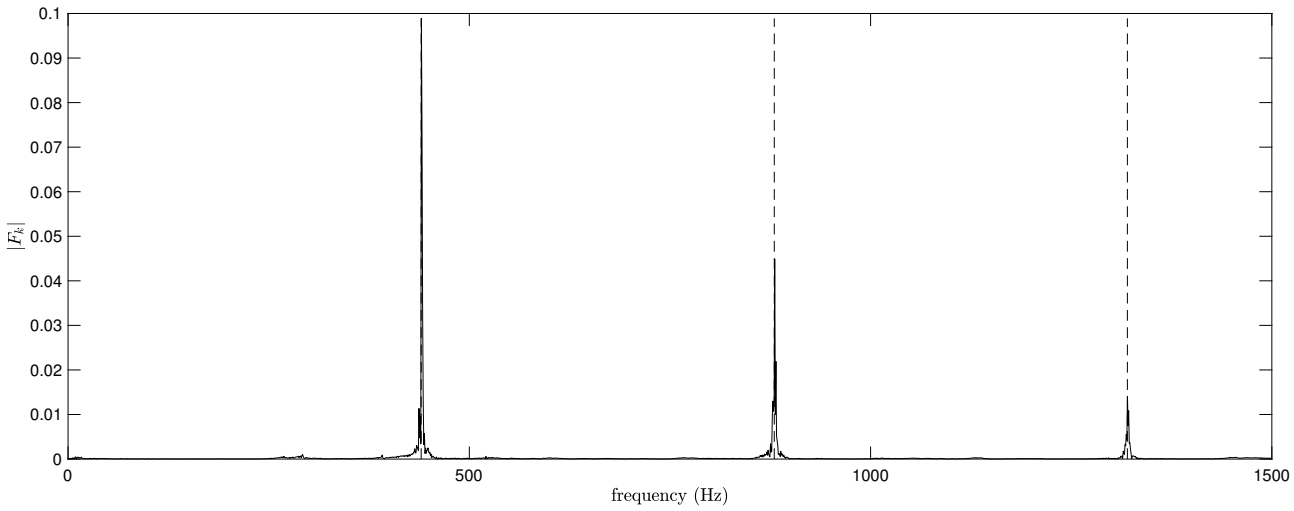


Figure 13.5: Spectrum of an A note on a real violin. The dashed lines are indicating the frequencies of 440 Hz, 880 Hz, and 1320 Hz.

where the note is played. What if we want a measure of instantaneous frequency at every moment in time? This can be constructed through the idea of windowing: as the note is played, construct a window of a certain interval, say $[t_0 - \Delta t, t_0 + \Delta t]$ of time. Compute the DFT in this window. Move the window to the right (in increasing t_0) and repeat the procedure. What results is a spectral plot for each window. This can be visualized as a spectrogram shown in Fig. 13.6.

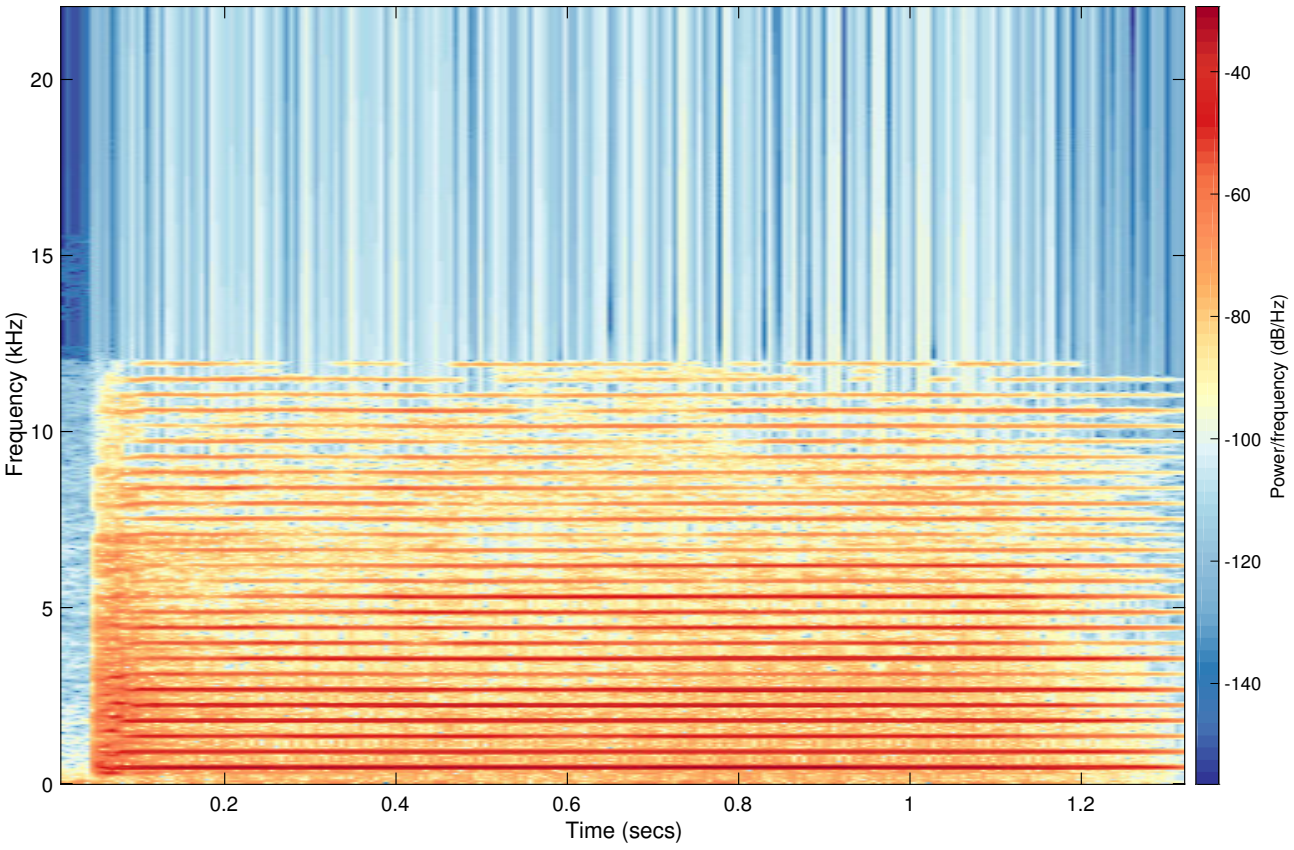


Figure 13.6: Spectogram of the violin A note constructed using Matlab's `spectrogram` function.

The horizontal and vertical axes are t and frequency (in kHz), respectively. The colour indicates the magnitude of the respective frequency component (or power).

Chapter 14

Terminology of PDEs

This chapter is mainly about terminology. We will introduce the following terms: (i) partial differential equation (PDE); (ii) linear PDE; (iii) homogeneous PDE; (iv) boundary-value problem (BVP); (v) Dirichlet and Neumann conditions.

14.1 Terminology

Notation 14.1. We often will write the partial derivative as a subscript.

$$\frac{\partial u}{\partial x} = u_x, \qquad \frac{\partial^2 u}{\partial x^2} = u_{xx}, \qquad \frac{\partial^2 u}{\partial x \partial t} = u_{xt}$$

Note that in this unit we will always assume that all functions are nice enough for their partial derivatives to commute (i.e. $u_{xt} = u_{tx}$).

Definition 14.2 (PDE)**.** A PDE for $u(x,t)$ is an equation of the form

$$F(x,t,u,u_x,u_t,u_{xx},u_{xt},\ldots) = 0. \tag{14.1}$$

The variables x,t are called the independent variables. If $u(x,t)$ satisfies (14.1), we say that u is a solution to the PDE.

Definition 14.3 (Order)**.** A PDE is *nth order* if it only involves the function and its derivatives up to and including derivatives of order n.

Example 14.4 (First order PDEs)**.** A first order PDE for $u(x,t)$ involves u, u_x, u_t. Some famous first-order PDEs are

$u_x - u_t = 0$	One-way wave equation
$u_x^2 + u_y^2 = 1$	Eikonal equation
$u_t + u\,u_x = 0$	Inviscid Burgers' equation

Example 14.5 (Second-order PDEs)**.** Some famous second-order PDEs are

$u_{tt} - u_{xx} = 0$	Wave equation
$u_t - u_{xx} = 0$	Heat equation
$u_{xx} + u_{yy} = 0$	Laplace's equation
$u_{xx} + u_{yy} = f$	Poisson's equation
$\mathrm{i}u_t + u_{xx} + V(x,t)u = 0$	Schrödinger's equation
$u_{xx}u_{yy} - u_{xy}^2 = 0$	Monge-Ampère equation

Definition 14.6 (Linear)**.** Recall that an operator L (acting on functions) is *linear* if

$$L(\lambda u + \mu v) = \lambda L(u) + \mu L(v)$$

for any two functions u and v, and arbitrary constants λ and μ.

Definition 14.7 (Linear PDE)**.** A PDE is *linear* if it can be written as

$$L(u) = f \tag{14.2}$$

where L is a linear differential operator and f is known.

Note that all the equations in Examples 14.4 and 14.5 are linear except for the Eikonal equation, the inviscid Burger's equation, and the Monge-Ampère equation.

Definition 14.8 (Homogeneous PDE)**.** A linear PDE is *homogeneous* if $f = 0$ in (14.2), i.e. there is no term that is independent of u and its derivatives.

Note that (i) a linear homogeneous equation has zero as one solution; and (ii) all the linear equations in Examples 14.4 and 14.5 are homogeneous apart from Poisson's equation.

Lemma 14.9 (Principle of Superposition)**.** If u and v are two solutions to a homogeneous linear PDE, then $\lambda u + \mu v$ is also a solution for any constants λ and μ.

Proof. The proof follows immediately from linearity and homogeneity. We have that $L(u) = 0$ and $L(v) = 0$. By linearity $L(\lambda u + \mu v) = \lambda L(u) + \mu L(v)$; therefore $L(\lambda u + \mu v) = 0$.

Example 14.10 (Linearity of Heat equation)**.** We have already encountered a homogeneous linear PDE given as the 1D heat equation of Chapter 10,

$$u_t = \kappa u_{xx}. \tag{14.3}$$

Notice that indeed this can be written in the form of $L(u) = f$ in (14.2) if we set

$$L = \partial_t - \kappa \partial_{xx}, \tag{14.4}$$

and where $f \equiv 0$.

14.2 Boundary-value problems

One usually meets PDEs as part of a *boundary value problem.*

Definition 14.11 (Boundary value problem (BVP))**.** A boundary value problem (BVP) for a PDE has three ingredients:

1. a PDE;
2. a set in which it is posed, we will call this the *domain*;
3. some conditions that the solution must satisfy on the boundary of the domain ("boundary conditions").

If the PDE involves a time variable t, then it is natural to call the boundary conditions that correspond to specifying the initial values of u and its derivatives *initial conditions.* Depending on the particular domain, the problem might then be called an *initial-value problem* (or even an *initial-boundary value problem*).

Remark 14.12 (Initial/boundary conditions for ODEs)**.** Recall that the solution of the 1st order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay$$

(where a is constant) has one arbitrary constant, and therefore we need one condition, for example $y(0) = c_1$, to determine the solution uniquely. If the t variable corresponds to time, the ODE is posed for $t \geq 0$, and the condition is given at $t = 0$ (as above), then it is natural to call this condition an "initial condition".

Recall that the solution of the 2nd order ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = ay \tag{14.5}$$

(where a is constant) has two arbitrary constants, and therefore we need two conditions to fix them, for example $y(0) = c_1$ and $y'(0) = c_2$. Again, if the t variable corresponds to time, the ODE is posed for $t \geq 0$, and both conditions are at $t = 0$, then it is natural to call these conditions initial conditions.

When the independent variable corresponds to space and not time, one often encounters the ODE (14.5) posed on a finite interval, i.e. the ODE is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = ay \quad \text{for} \quad x \in [0, L]. \tag{14.6}$$

It is natural to impose one condition at each endpoint $x = 0$ and $x = L$ (the boundary of the domain), and one usually imposes the value of *either* y *or* y' (if you impose y'' then this is the same as prescribing y since $y'' = ay$ by the ODE). For instance, typical boundary conditions for (14.6) may include specifying values for $\{y(0), y(L)\}$, or $\{y'(0), y'(L)\}$, or $\{y(0), y'(L)\}$, and so forth.

Definition 14.13 (Linear Dirichlet, Neumann, Robin conditions)**.** There are three types of BVPs that arise in most applications. If the value of the solution is specified along the boundary of the domain, this is called a Dirichlet condition. If the normal derivative is specified along the boundary, this is called a Neumann condition. If a combination of the function value and its normal derivative is specified along the boundary, this is called a mixed or Robin condition.

More specifically, consider a boundary-value problem defined on a domain D with boundary values on the boundary, ∂D. Let ∂D_i be a segment of the entire boundary. For a given function, h, a Dirichlet condition on ∂D_i is

$$u(\boldsymbol{x}) = h(\boldsymbol{x}) \quad \text{for } x \in \partial D_i. \tag{14.7a}$$

A Neumann condition is

$$\frac{\partial u(\boldsymbol{x})}{\partial n} = h(\boldsymbol{x}) \quad \text{for } x \in \partial D_i. \tag{14.7b}$$

A mixed condition (with constant coefficients) is

$$\alpha u(\boldsymbol{x}) + \beta \frac{\partial u(\boldsymbol{x})}{\partial n} = h(\boldsymbol{x}) \quad \text{for } x \in \partial D_i, \tag{14.7c}$$

where α and β are constant.

Example 14.14. Consider the ordinary BVP (14.6). Here, the domain is $[0, L]$, so the boundary is $x = 0$ and $x = L$. For the left boundary, $x = 0$, an example of a Dirichlet boundary condition is

$$y(0) = C, \tag{14.8a}$$

for a constant C. An example of a Neumann condition is

$$y'(0) = C, \tag{14.8b}$$

and an example of a mixed condition is

$$y(0) + y'(0) = C. \tag{14.8c}$$

Definition 14.15 (Homogeneous boundary conditions)**.** For the three examples of Dirichlet, Neumann, and mixed boundary conditions in (14.7), if $h \equiv 0$, then the boundary condition is considered to be homogeneous. Otherwise, the boundary condition is inhomogeneous.

Definition 14.16 (1D heat equation with zero Dirichlet boundary conditions and prescribed initial condition)**.** In Sec. 10.4, we derived the heat distribution of a bar on $x \in [0, L]$ with the two ends at a fixed and zero temperature, and with initial temperature distribution $f(x)$. This yields the system

$$u_t = \kappa u_{xx}, \tag{14.9}$$

$$u(0, t) = 0 = u(L, t), \tag{14.10}$$

$$u(x, 0) = f(x). \tag{14.11}$$

Indeed, all three boundary conditions imposed on the boundaries $x = 0$, $x = L$, and $t = 0$, specify the value of the solution, u. The boundary conditions on $x = 0$ and $x = L$ are Dirichlet and homogeneous. The initial condition on $t = 0$ is inhomogeneous.

Chapter 15

Soluton of the heat equation using separation of variables

Let us take a moment to recap what we have done in the previous few chapters, beginning from Chapter 10. In Chapter 10, we showed how partial differential equations could be derived from physical laws. We showed that by applying conservation of energy to a one-dimensional bar on $x \in [0, L]$, the temperature along the bar is described by eqn (10.10), or

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \tag{15.1}$$

where $\kappa = k/(\rho c)$ is the thermal diffusivity, and $u = u(x, t)$ is the temperature. Next (in the terminology of Chapter 14) we defined the homogeneous Dirichlet problem

$$u_t = \kappa u_{xx}, \tag{15.2a}$$
$$u(0, t) = 0 = u(L, t), \tag{15.2b}$$
$$u(x, 0) = f(x), \tag{15.2c}$$

which models the heat distribution in the 1D bar when the temperature at the ends is fixed at zero and the bar begins at a given initial temperature, $f(x)$.

At this point, we showed that a general solution could be constructed to the above problem through separation of variables. Writing

$$u(x, t) = X(x)G(t), \tag{15.3}$$

we derived solutions that satisfied the boundary conditions (15.2b), but we could not satisfy the initial condition (15.2c) for this involves the notion of a Fourier series, which was then the subject of Chapters 11 and 12.

In this chapter, we return to the separation of variables argument and finally complete the problem.

15.1 Fourier series solution for heat problem (15.2)

This section contains some repetition as we repeat the derivation presented earlier in Sec. 10.4. Again, we write the solution of system (15.2) in the separated form of (15.3). In this case, substitution into the PDE and re-arranging all functions of time to the left and functions of x to the right gives

$$\frac{G'(t)}{\kappa G(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \leq 0. \tag{15.4}$$

In the above equation, the left hand-side is entirely a function of time while the right hand-side is a function of x. This means that both sides must be equal to a constant, and we have set this constant

to be zero or negative. The fact that the constant cannot be positive can be verified *a posteriori*. This yields now two ODEs:

$$G'(t) = -\lambda^2 \kappa G(t) \tag{15.5}$$
$$X''(x) = -\lambda^2 X(x). \tag{15.6}$$

We now seek to solve for G and X. Firstly, if $\lambda = 0$, then we conclude that $G(t) \equiv$ const. and $X(x) = Ax + B$. However, application of the boundary implies that in order for G not to be trivial,

$$u(0,t) = 0 \Rightarrow X(0)G(t) = 0 \Rightarrow X(0) = 0 \tag{15.7a}$$
$$u(L,t) = 0 \Rightarrow X(L)G(t) = 0 \Rightarrow X(L) = 0. \tag{15.7b}$$

Thus $X(x) = Ax + B$ would imply that $A = 0$ and $B = 0$, producing the trivial solution. We consider therefore the case that $\lambda \neq 0$. Then

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x), \tag{15.8}$$
$$G(t) = D\mathrm{e}^{-\kappa\lambda^2 t}. \tag{15.9}$$

The first boundary condition in (15.7) tells us that $A = 0$ and the second tells us that

$$B\sin(\lambda L) = 0 \Rightarrow \lambda_n = \frac{\pi n}{L}, \quad n = 1, 2, 3, \ldots. \tag{15.10}$$

Notice that $n = 0$ is trivial, and moreover, negative integers, n simply lead to negating the arbitrary constant. Each value of λ_n leads to a possible solution. In general, we might consider adding together all the possible components in order to get a general form:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\kappa\lambda_n^2 t}\sin(\lambda_n x), \tag{15.11}$$

where $\lambda_n = \pi n/L$.

Now the last question we are left with is how to impose the initial condition, (10.20). If we substitute $t = 0$ into (15.11), we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n x}{L}\right). \tag{15.12}$$

By now, we understand that (15.12) is a Fourier sine series of $f(x)$ on the interval $x \in [0, L]$. Thus, based on Theorem 12.7, we can write down the coefficients, B_n, of (15.12). These are given by forming the odd-periodic extension of $f(x)$, originally from $[0, L]$ to $[-L, L]$. Then they are given by (12.15), and hence

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{15.13}$$

and indeed this completes the problem.

Theorem 15.1 (Solution of 1D heat equation with zero Dirichlet boundary conditions and prescribed initial condition)**.** The solution of system 15.2 or alternatively Definition 14.16 is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\kappa\lambda_n^2 t}\sin(\lambda_n x), \tag{15.14a}$$
$$\text{where } \lambda_n = \frac{\pi n}{L} \quad \text{and} \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{15.14b}$$

15.2 Summary of separation of variables procedure

Algorithm 15.2 (Separation of variables algorithm)**.**

0. Reduce the BVP to one with zero (homogeneous) boundary conditions.
1. Assume that $u(x,t) = X(x)\,G(t)$, substitute into the PDE, and obtain separated ODEs for $X(x)$ and $G(t)$.
2. Solve the ODEs for $X(x)$ and $G(t)$.
3. Impose BCs to obtain a family of solutions depending on $n \in \mathbb{Z}$.
4. Use the Principle of Superposition to find a solution as a sum over $n \in \mathbb{Z}$.
5. Fix the unknown coefficients in the sum by applying the initial conditions.

Notice that in our study of the homogeneous Dirichlet problem of (15.2), we did not need Step 0, as the problem already has zero boundary conditions. This step is included for the more general boundary conditions to come.

15.3 Solution of the 1D inhomogeneous heat equation

Definition 15.3 (1D heat equation with inhomogeneous Dirichlet boundary conditions and prescribed initial condition)**.** In Sec. 10.4, we derived the heat distribution of a bar on $x \in [0, L]$ with the two ends at a fixed and constant temperature, and with initial temperature distribution $f(x)$. This yields the system

$$u_t = \kappa u_{xx} \quad \text{on } x \in [0, L], \tag{15.15a}$$
$$u(0,t) = T_0 \quad \text{and} \quad u(L,t) = T_1, \tag{15.15b}$$
$$u(x,0) = f(x). \tag{15.15c}$$

In the case that $T_0 = 0 = T_1$, this reduces to the homogeneous Dirichlet problem.

The procedure to solve the inhomogeneous Dirichlet problem in Definition 15.3 requires only a single sub-step in order to convert it to a homogeneous Dirichlet problem. We essentially write the solution broken into two parts: one part, $U(x)$, that satisfies the time-independent heat equation, and the other part, $\widehat{u}(x,t)$ that will produce the correct homogeneous Dirichlet conditions. Thus, write:

$$u(x,t) = U(x) + \widehat{u}(x,t). \tag{15.16}$$

Now we require that $U(x)$ satisfies the time-independent heat equation. Since $U(x)$ has no dependence on t, then $U_t = 0$. Consequently, we impose that $U(x)$ satisfies

$$0 = \kappa U_{xx}, \tag{15.17a}$$
$$U(0) = T_0 \quad \text{and} \quad U(L) = T_1. \tag{15.17b}$$

Solving this gives

$$U(x) = T_0 + \frac{(T_1 - T_0)}{L} x. \tag{15.18}$$

Finally, we substitute (15.16) into (15.15) and this yields

$$\widehat{u}_t = \kappa \widehat{u}_{xx}, \tag{15.19a}$$
$$\widehat{u}(0,t) = 0 \quad \text{and} \quad \widehat{u}(L,t) = 0, \tag{15.19b}$$
$$\widehat{u}(x,0) = -U(x) + f(x). \tag{15.19c}$$

We are now done, as (15.19) is a homogeneous Dirichlet problem, and can thus be solved using the procedure of Algorithm 15.2.

Example 15.4. Consider the inhomogeneous Dirichlet problem with $L = \pi$, $\kappa = 1$, $T_0 = 2$, $T_1 = 1$, and $u_0(x) = 0$ for $0 < x < 2$. That is, solve

$$u_t = u_{xx} \quad \text{on } x \in [0, \pi], \tag{15.20a}$$
$$u(0, t) = 2 \quad \text{and} \quad u(\pi, t) = 1, \tag{15.20b}$$
$$u(x, 0) = 0. \tag{15.20c}$$

We follow the procedure of Sec. 15.3. First, we solve the time-independent problem for $U(x)$ in system (15.17). Thus we solve

$$U'' = 0, \quad U(0) = 2, \quad U(\pi) = 1. \tag{15.21}$$

This gives

$$U(x) = 2 - \frac{x}{\pi}. \tag{15.22}$$

Next, we write the solution, $u(x, t)$, so as to include the time-independent portion. Thus, we write

$$u(x, t) = U(x) + \widehat{u}(x, t).$$

Now, $\widehat{u}$ satisfies the homogeneous Dirichlet problem with

$$\widehat{u}_t = \widehat{u}_{xx}, \tag{15.23a}$$
$$\widehat{u}(0, t) = 0 \quad \text{and} \quad \widehat{u}(\pi, t) = 0, \tag{15.23b}$$
$$\widehat{u}(x, 0) = -U(x) + 0. \tag{15.23c}$$

We have already solved the homogeneous Dirichlet problem in Sec. 15.1. The solution is given by

$$\widehat{u}(x, t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-n^2 t} \sin(nx), \tag{15.24}$$

where

$$B_n = -\frac{2}{\pi} \int_0^{\pi} \Big(2 - \frac{x}{\pi}\Big) \sin(nx) \, \mathrm{d}x. \tag{15.25}$$

After integrating by parts, we find that

$$B_n = \frac{2}{n\pi}\Big((-1)^n - 2\Big),$$

so

$$u(x, t) = \Big(2 - \frac{x}{\pi}\Big) + \sum_{n=1}^{\infty} \frac{2}{n\pi}\Big((-1)^n - 2\Big) \exp\left(-n^2 t\right) \sin(nx). \tag{15.26}$$

Figure 15.1 shows the solution plotted at various times (by summing the first 199 terms of the series). These sequence of graphs demonstrate why the function $U(x)$ in (15.22) called the *steady state or time-independent solution.* As $t \to \infty$, all the terms in the series in (15.26) tend to zero (since $\exp(-n^2 t) \to 0$ as $t \to \infty$), and thus $u(x, t) \to 2 - x/\pi$.

15.4 The Neumann problem

The inhomogeneous Neumann problem for the heat equation is identical to the inhomogeneous Dirichlet problem, except that the boundary conditions are replaced by conditions on the derivatives:

$$u_x(0, t) = F_0 \quad \text{and} \quad u_x(L, t) = F_1, \tag{15.27}$$

where F_0, F_1 are constants.

Note that when the heat equation is used to model temperature in a bar, the Neumann boundary conditions (15.27) correspond to specifying that the flux of heat through the two ends of the bar is F_0 and F_1 respectively.

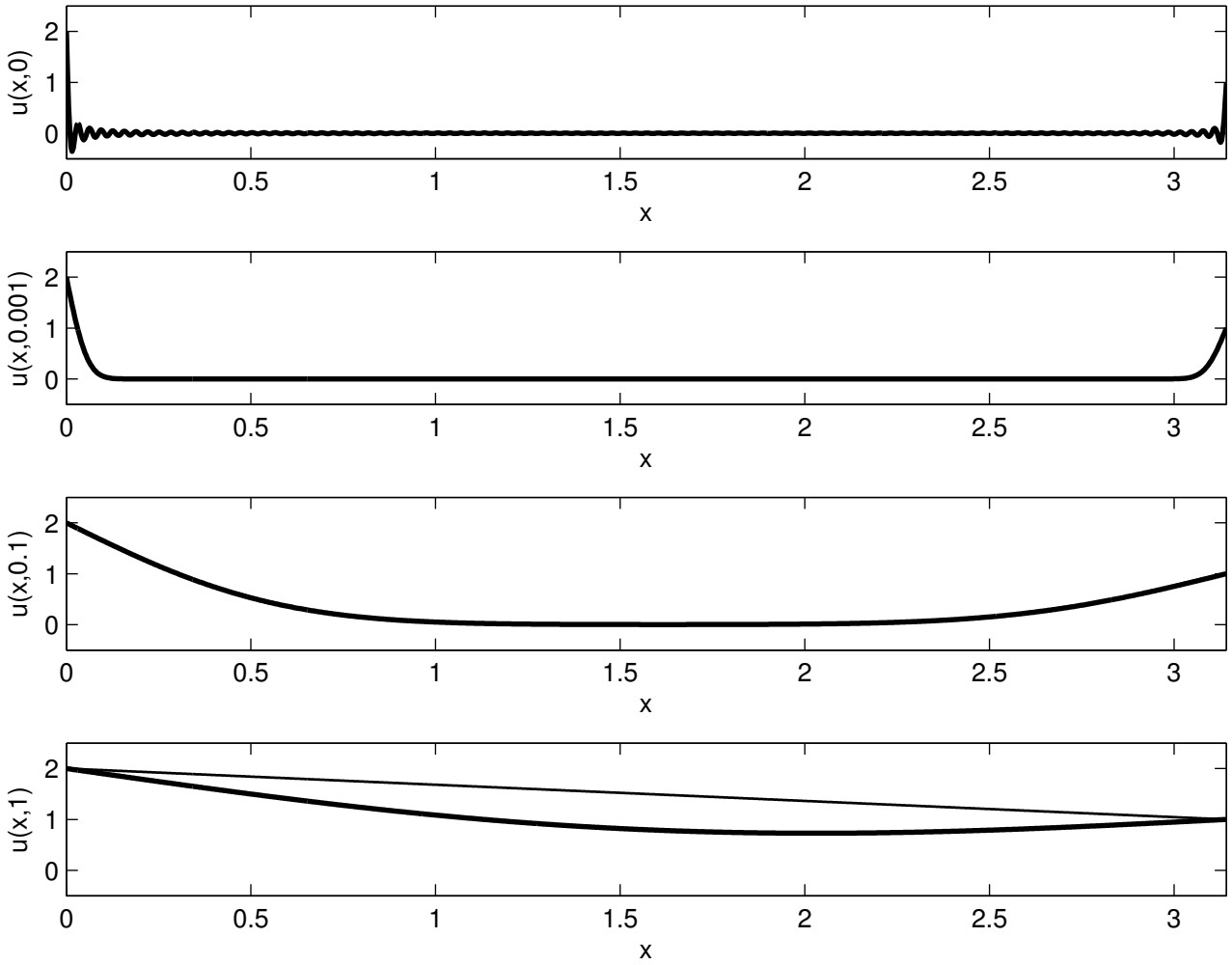


Figure 15.1: The solution to inhomogeneous Dirichlet problem for the heat equation in Example 15.4, plotted at $t = 0$, $t = 0.001$, $t = 0.1$, and $t = 1$. In the figure for $t = 1$, the function $2 - x/\pi$ (the steady state solution) is shown with a thin line. Note the Gibbs oscillations in the limit $t \to 0$; this is because of the jump in the initial/boundary conditions at $t = 0$ (see the diagram you drew).

Chapter 16

Wave equation using separation of variables

The wave equation is given by

$$u_{tt} = c^2 u_{xx}, \tag{16.1}$$

and we encountered its physical derivation in Chapter 10. This equation models the transverse vibration of a finite string in $x \in [0, L]$, where $u(x, t)$ corresponds to the transverse position of the string (the height), and where $c = \sqrt{T/\rho}$ is known as the wave speed, and depends on the tension, T, and density ρ in the string.

16.1 The Dirichlet and Neumann problems for a finite string

Like in our examination of the heat equation in the previous chapter, we are interested in studying the boundary-value problem (BVP) that consists of the wave equation on a finite interval, $x \in [0, L]$, and with time, $t > 0$. In addition to the initial condition that indicates the beginning displacement, $u(x, 0)$, and velocity, $u_t(x, 0)$ of the wave, we must again specify the boundary conditions at $x = 0$ and $x = L$.

The case of the zero Dirichlet problem corresponds to fixing the ends of the string at zero height.

Definition 16.1 (1D wave equation with homogeneous Dirichlet boundary conditions and prescribed initial conditions)**.** Consider the wave equation on a string $x \in [0, L]$ with the two ends clamped at zero height, and with initial displacement given by $u_0(x)$ and initial velocity by $v_0(x)$. This yields the BVP

$$u_{tt} = c^2 u_{xx}, \tag{16.2a}$$
$$u(0, t) = 0 = u(L, t), \tag{16.2b}$$
$$u(x, 0) = u_0(x), \tag{16.2c}$$
$$u_t(x, 0) = v_0(x), \tag{16.2d}$$

applied for $t > 0$.

The Neumann problem for the wave equation is identical to the Dirichlet problem in Definition 16.1, except that (16.2b) is replaced by

$$u_x(0, t) = 0 = u_x(L, t). \tag{16.3}$$

For waves on a string, the Neumann boundary conditions correspond to the string being allowed to slide freely in the vertical direction at each end (imagine the string being attached to a vertical rod via a movable loop).

16.2 Solution of the Dirichlet problem using separation of variables

We now solve the Dirichlet problem (Definition 16.1) using separation of variables is solved similarly to the solution of the heat equation from the previous chapter and uses Algorithm 15.2.

Assume that $u(x,t)$ can be written as

$$u(x,t) = X(x)\,G(t). \tag{16.4}$$

Then by the wave equation (16.2a), we have

$$X(x)\,G''(t) = c^2 X''(x)\,G(t),$$

or after re-arranging,

$$\frac{G''(t)}{G(t)} = c^2 \frac{X''(x)}{X(x)}. \tag{16.5}$$

Since (i) the left-hand side of the equation is a function of t only and the right-hand side is a function of x only, and (ii) t and x are independent variables, the only possibility is that both sides are equal to a constant. We thus set

$$\frac{G''(t)}{G(t)} = c^2 \frac{X''(x)}{X(x)} = -\lambda^2 < 0. \tag{16.6}$$

Again, like the situation of the heat equation, the constant should be assumed to be negative, but this can only be seen *a posteriori* (see for example, Problem Set 8, Q4). In particular, the assumption of a positive constant produces a trivial solution after the boundary conditions are applied. Thus we have the pair of *separate* ODEs for X and G:

$$G''(t) + \lambda G(t) = 0 \quad \text{and} \quad X''(x) + (\lambda/c)^2 X(x) = 0. \tag{16.7}$$

Solving these ODEs, we have

$$\begin{aligned} G(t) &= A\cos(\lambda t) + B\sin(\lambda t) \\ X(x) &= C\cos(\lambda x/c) + D\sin(\lambda x/c) \end{aligned} \tag{16.8}$$

Thus, our separable solution to (16.4) to the wave equation now reads

$$u(x,t) = \Big[C\cos(\lambda x/c) + D\sin(\lambda x/c)\Big]\Big[A\cos(\lambda t) + B\sin(\lambda t)\Big]. \tag{16.9}$$

We now impose the Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$ for all $t > 0$. These imply that

$$X(0)T(t) = X(L)T(t) = 0 \quad \text{for} \quad t > 0,$$

and the only way to satisfy this with a non-zero $T(t)$ is to have

$$X(0) = X(L) = 0.$$

From the equation for $X(x)$ in (16.8), we see that $X(0) = C$ and thus we need to take $C = 0$. With this choice of C, the condition $X(L) = 0$ becomes $D\sin(\lambda L/c) = 0$. If D were equal to zero, then $X(x)$ (and hence $u(x,t)$) would be zero for all x, so we therefore need $\sin(\lambda L/c) = 0$. Thus we conclude,

$$\sin(\lambda L/c) = 0 \implies \frac{\lambda L}{c} = n\pi \implies \lambda_n = \frac{n\pi c}{L} \quad \text{for} \quad n \in \mathbb{Z},$$

where we have indexed each respective λ with λ_n. Thus we conclude that the separable solution is given by

$$u(x,t) = D\sin\left(\frac{n\pi x}{L}\right)\left(A\cos\left(\frac{n\pi ct}{L}\right) + B\sin\left(\frac{n\pi ct}{L}\right)\right), \quad n \in \mathbb{Z}.$$

Notice that $DA = A_n$ and $DB = B_n$ are again arbitrary constants. Hence we can re-write this last expression as

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \quad \text{for } n \in \mathbb{Z}^+ = \{1,2,3,\ldots\}. \tag{16.10}$$

Like for the case of the heat equation, we notice that the negative integers, $n = -1, -2, -3, \ldots$ correspond to negating the arbitrary constants A_n and B_n (since the sine function is odd), and hence can be dropped. The $n = 0$ case yields $u_n = 0$ so is trivial. Thus (16.10) forms all the possible separable solutions, each indexed with a respective positive integer n. The individual solutions, $u_n(x,t)$, are often called the *eigenfunctions* with corresponding *eigenvalues*, λ_n.

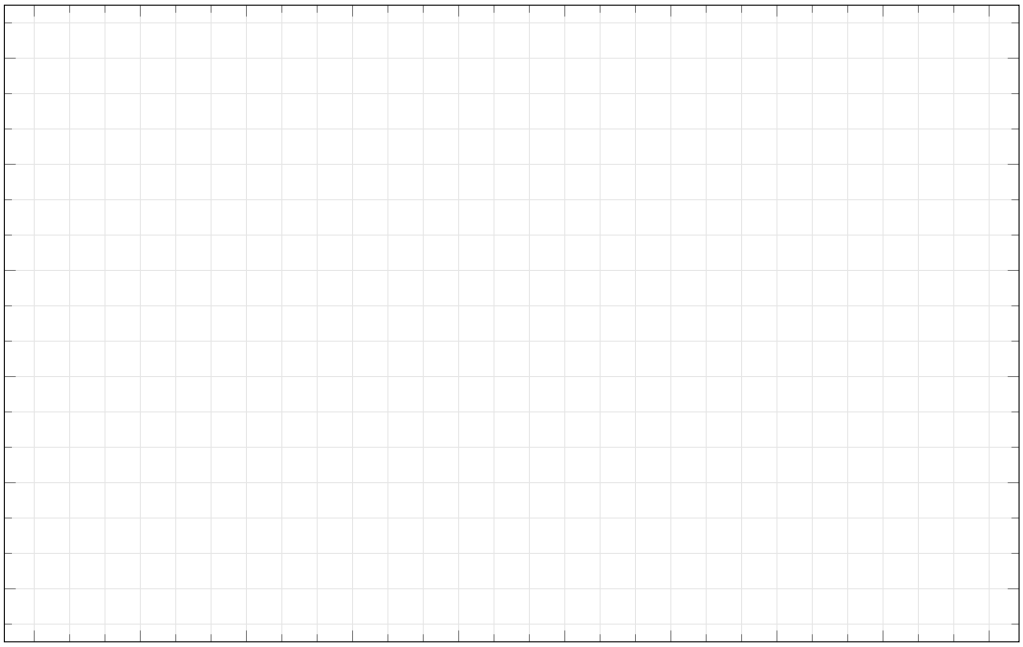


Figure 16.1: Graph of the fundamental mode of a string, i.e. $u(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right)$ at $t = 0, L/c, 2L/c$.

Example 16.2 (Examination of $u_n(x,t)$ in eqn (16.10))**.** Physically, what do the individual separable solutions, u_n, correspond to?

Solution. For simplicity, we consider a string length $L = \pi$. In (16.10), consider $n = 1$ and $\{A_1, B_1\} = \{1, 0\}$. Then we have

$$u_1(x,t) = \sin(x)\cos(ct). \tag{16.11}$$

Recall that the function $\cos(kt)$ has period $2\pi/k$. Thus note that $\cos(ct)$ has period $T = 2\pi/c$. Consequently at times $t = 0$ and $t = T$, $\cos(0) = \cos(2\pi) = 1$ and the evolution of u_1 returns to its original position, where

$$u_1(x,\,0) = u_1(x,\,T) = \sin(x).$$

Also, half-way through the temporal period, at $t = T/2$, we have $\cos(\pi) = -1$ and

$$u(x, T/2) = -\sin(x).$$

Figure 16.1 shows the solution (16.11) plotted at $t = 0,\, T/2,\, T$. This is a standing wave with a single anti-node within $[0, \pi]$.

Consider now the second mode, with $n = 2$. Again, if $A_2 = 1$, $B_2 = 0$ then (16.10) yields

$$u_2(x,t) = \sin(2x)\cos(2ct). \tag{16.12}$$

This time, notice that the temporal period for $\cos(2ct)$ is $T = \pi/c$. Thus, when $t = 0$ and $t = T$,

$$u_2(x, 0) = u_2(x, T) = \sin(2x).$$

Similarly, at half the period, $t = T/2$,

$$u_2(x, T/2) = -\sin(2x).$$

In words, then the $n = 2$ mode is a standing wave solution with two antinodes. The pattern continues.

Implementation of Fourier series

Returning to our separated solution (16.10), we recall that our goal is to solve the Dirichlet problem (Definition 16.1). We have already made sure our solution in (16.10) satisfies the wave equation (16.1) and Dirichlet boundary conditions (16.2b). It remains to satisfy the initial conditions (16.2c) and (16.2d).

Like for the situation of the heat equation, with u_n given by (16.10), we have for the initial displacement and velocity,

$$u_n(x, 0) = A_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \partial_t u_n(x, 0) = \left(\frac{n\pi c}{L}\right) B_n \sin\left(\frac{n\pi x}{L}\right).$$

Thus, unless the specified $u_0(x)$ and $v_0(x)$ take these very special forms, our u_n given by (16.10) cannot individually satisfy the the most general initial conditions of $u(x, 0) = u_0(x)$, $u_t(x, 0) = v_0(x)$. Given a geneal displacement and velocity for which the string must satisfy at $t = 0$, we would expect that the solution must be formed through a linear combination of our u_n solutions—and hence a Fourier series.

We now use the fact that the wave equation is linear, and therefore by the Principle of Superposition (Lemma 14.9),

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \tag{16.13}$$

is also solution of the wave equation.

We now use our knowledge of Fourier series to chose A_n and B_n so that (16.13) satisfies the initial conditions (16.2c) and (16.2d). Firstly, setting $t = 0$ in (16.13) gives

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \tag{16.14}$$

and we need this to be equal to the prescribed function $u_0(x)$. We recognize this as a Fourier sine series, which arises through odd functions.

We thus begin with $u_0(x)$ on $(0, L)$ and produce the odd extension to $(-L, L)$. This forms the odd $2L$-periodic extension. We then calculate its Fourier sine series as in Sec. 12.3. Using the odd version of Theorem 12.7, we recall that an odd function, f, defined on $(-L, L)$, can be written as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Therefore, if we let

$$A_n = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

With this choice of A_n, then (16.14) now satisfies the initial condition, $u(x, 0) = u_0(x)$.

It remains to satisfy the initial condition involving v_0. Differentiating (16.13) with respect to t and then setting $t = 0$, we find

$$v_0(x) = \sum_{n=1}^{\infty} \underbrace{\left[\left(\frac{n\pi c}{L}\right) B_n\right]}_{b_n} \sin\left(\frac{n\pi x}{L}\right). \tag{16.15}$$

Again, we can construct a sine series for the odd $2L$-periodic extension of v_0, and it then follows that

$$b_n = \frac{2}{L} \int_0^L v_0(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x,$$

and thus

$$b_n = \frac{n\pi c}{L} B_n \quad \text{i.e.} \quad B_n = \frac{L}{n\pi c} b_n.$$

In summary, we have proved the following result.

Theorem 16.3 (Solution of the Dirichlet problem via separation of variables)**.** The expression

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \tag{16.16}$$

satisfies the Dirichlet problem (Definition 16.1) if

$$A_n = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x, \tag{16.17}$$

$$B_n = \frac{2}{L} \frac{L}{n\pi c} \int_0^L v_0(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x. \tag{16.18}$$

Example 16.4 (Plucked string)**.** Consider the Dirichlet problem with $L = \pi$, $c = 1$, and the two initial conditions

$$u_0(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} < x < \pi \end{cases} \quad \text{and} \quad v_0 = 0.$$

Solution. Note that this corresponds to a string of length π, plucked at its midpoint, and released with zero velocity. Since $v_0 = 0$,

$$B_n = 0 \quad \text{for all } n.$$

Using the definition of u_0 in (16.17) we have

$$A_n = \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} x \sin(nx) \, \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin(nx) \, \mathrm{d}x\right]$$

We now need to do some integration. We note first that the change of variables $y = \pi - x$ shows that

$$\int_{\pi/2}^{\pi} (\pi - x) \sin(nx) \, \mathrm{d}x = -(-1)^n \int_0^{\pi/2} y \sin(ny) \, \mathrm{d}y,$$

so if n is even then $A_n = 0$, and when n is odd

$$A_n = \frac{4}{\pi} \int_0^{\pi/2} x \sin(nx) \, \mathrm{d}x.$$

Using integration by parts we find that

$$A_n = \frac{4}{\pi n^2} \sin\left(\frac{n\pi}{2}\right)$$

when n is odd, and in fact this formula is true when n is even (since $\sin(m\pi) = 0$). Therefore, Theorem 16.3 implies that the solution to this particular Dirichlet problem is given by

$$u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) \frac{1}{n^2} \sin(nx) \cos(nt).$$

Figure 16.2 shows this solution plotted at various times.

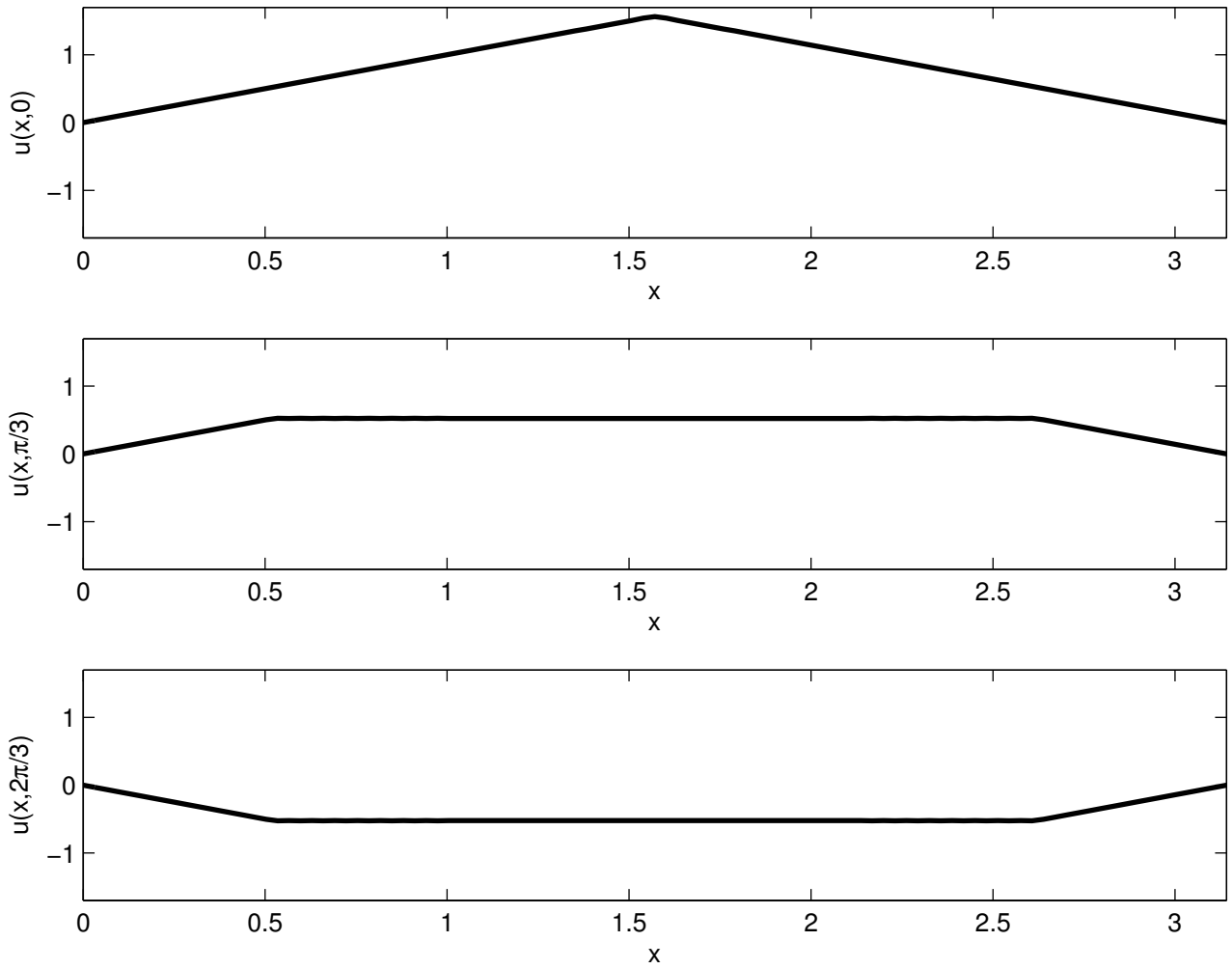


Figure 16.2: The plucked string from Example 16.4 at $t = 0, \pi/3, 2\pi/3$.

Chapter 17

Laplace's equation using separation of variables

The last of the three 'holy trinity' equations is Laplace's equation. Like the heat and wave equations, Laplace's equation is a second-order linear PDE. We had already encountered the equation in the context of the two-dimensional heat equation[1],

$$u_t = \kappa \nabla^2 u = \kappa(u_{xx} + u_{yy}) \quad \text{for } (x, y) \in D, \tag{17.1}$$

which describes the temperature, $u(x, y, t)$, on a planar region $D \subseteq \mathbb{R}^2$. In the limit of long time, $t \to \infty$, it is expected that the steady-state solution is time-independent, and thus $\partial u/\partial t = 0$. Thus, the steady-state heat distribution is described by the equation,

$$\nabla^2 u = u_{xx} + u_{yy} = 0. \tag{17.2}$$

This is the two-dimensional Laplace's equation. Note that the symbols ∇^2 and Δ are both used as shorthand for the differential operator

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and its higher dimensional analogues. In this course, we use ∇^2.

In fact, Laplace's equation connected with a vast range of physical applications, ranging from the study of electromagnetism (in Maxwell's equations), astronomy (in connection with gravitational potentials), and fluid and solid dynamics (*e.g.* the Navier-Stokes equations, the heat equation, the wave equation).

17.1 BVPs for Laplace's equation

The (inhomogeneous) Dirichlet problem for Laplace's equation is defined as follows.

Definition 17.1 (Dirichlet problem for Laplace's equation)**.** Given a domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$ and a function $f(x, y)$ defined for $(x, y) \in \partial\Omega$, find $u(x, y)$ in Ω such that

$$\nabla^2 u = 0 \quad \text{for} \quad (x, y) \in \Omega$$

and

$$u(x, y) = f(x, y) \quad \text{for} \quad (x, y) \in \partial\Omega. \tag{17.3}$$

[1] See also Q3 of PS4 where the equation is derived.

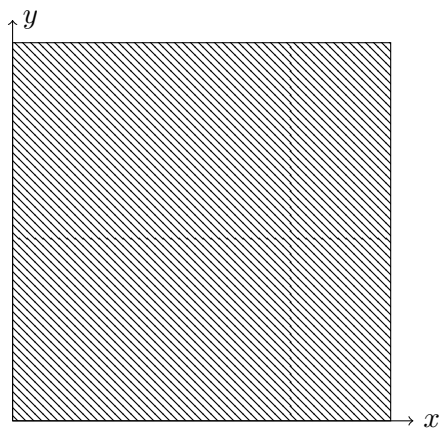


Figure 17.1: The Dirichlet boundary conditions we first consider for Laplace's equation.

In other words, u satisfies Laplace's equation on the interior of the domain, and is equal to a known $f(x, y)$ on the boundary. The Neumann problem is defined similarly, with (17.3) replaced by

$$\frac{\partial u}{\partial n}(x, y) = g(x, y) \quad \text{for} \quad (x, y) \in \partial\Omega,$$

for some given function g on $\partial\Omega$, where $\partial u/\partial n = \boldsymbol{n} \cdot \nabla u$, where $\boldsymbol{n}$ is the outward-pointing unit normal vector to $\partial\Omega$. Note that we could also have a BVP where Dirichlet conditions are posed on part of $\partial\Omega$, and Neumann conditions are posed on the rest of $\partial\Omega$.[2]

In this section of the unit, we will use separation of variables find an explicit expression for the solution of the Dirichlet problem when Ω is a square and Ω is a disc.

17.2 Solution of the Dirichlet problem in a square with one inhomogeneous side

We consider Laplace's equation in a square with side length L, i.e.

$$u_{xx} + u_{yy} = 0 \quad \text{for} \quad (x, y) \in (0, L) \times (0, L), \tag{17.4}$$

and we first consider the particular Dirichlet boundary conditions that correspond to zero values on three sides and a non-zero value on the fourth side:

$$u(0, y) = u(L, y) = u(x, L) = 0, \quad \text{and} \quad u(x, 0) = f(x); \tag{17.5}$$

see Figure 17.2. The method we use is applicable when the domain is a rectangle but for simplicity we consider the square.

We now follow the separation of variables algorithm.

Step 1. We assume that

$$u(x, y) = X(x)\, Y(y). \tag{17.6}$$

Substituting this into the equation (17.4), we find

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left-hand side is a function of x only, the right-hand side is a function of y only, therefore both sides must be equal to a constant. Similar arguments to those used for the heat and wave equations

[2]Note that the Neumann condition is specified in terms of the derivative with respect to the normal direction. For instance if the boundary is along the x axis, the Neumann condition would be on $\partial u/\partial y$, as $\boldsymbol{n} = (0, \pm 1)$.

can be used to show that the constant must be negative. Thus we set[3]

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k^2 < 0. \tag{17.7}$$

and hence obtain the two separate ODEs

$$X''(x) + k^2 X(x) = 0 \quad \text{and} \quad Y''(y) - k^2 Y(y) = 0.$$

Step 2. We solve the ODEs to find

$$X(x) = C\cos(kx) + D\sin(kx),$$
$$Y(y) = A\cosh(ky) + B\sinh(ky).$$

Step 3. The Dirichlet boundary conditions at $x = 0$ and $x = L$ imply that

$$X(0)Y(y) = X(L)Y(y) = 0 \quad \text{for all} \quad y \in [0, L] \tag{17.8}$$

The only way to satisfy (17.8) with a non-zero Y is to have

$$X(0) = X(L) = 0;$$

as before these conditions imply that $C = 0$ and $\sin(kL) = 0$, i.e. $k = n\pi/L$ for $n \in \mathbb{Z}$.

Our separated solution (17.6) then becomes

$$u(x, y) = \sin\left(\frac{n\pi x}{L}\right)\left[A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)\right], \quad n \in \mathbb{Z}.$$

We now need to impose the boundary conditions at $y = 0$ and $y = L$. We concentrate on the $y = L$ one first, since (from our experience with the wave and heat equations) we expect that we'll need to use the principle of superposition and Fourier series to satisfy the boundary condition $u(x, 0) = f(x)$.

The boundary condition $u = 0$ when $y = L$ implies that

$$A\cosh(n\pi) + B\sinh(n\pi) = 0.$$

We could then solve this for A in terms of B, say, but this is a bit messy. A slicker way is to realise that we could also write $Y(y)$ as

$$Y(y) = \widetilde{A}\cosh(k(L - y)) + \widetilde{B}\sinh(k(L - y)).$$

since $\cosh(k(L-y))$ and $\sinh(k(L-y))$ are also two linearly independent solutions of $Y''(y)-k^2Y(y) = 0$. The condition $Y(L) = 0$ then implies $\widetilde{A} = 0$ and thus we obtain that the separable solution given by

$$u(x, y) = \widetilde{B}\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi}{L}(L - y)\right), \quad n \in \mathbb{Z},$$

satisfies Laplace's equation, and the three zero Dirichlet boundary conditions in (17.5).

Step 4. The principle of superposition then implies that

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi}{L}(L - y)\right)$$

satisfies Laplace's equation and the three zero Dirichlet boundary conditions in (17.5) (where we have dropped the $\widetilde{\ }$ on the constants for convenience).

[3]There is a trick to remembering whether the constant in (17.7) should be negative or otherwise. In this question, $X(0) = 0 = X(\pi)$ and the typical function that exihibits two zeros as required is oscillatory. Had the constant been forced to be positive (or zero), the reader can verify that only the trivial solution results.

Step 5. Applying the boundary condition at $y = 0$, we see that we need

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh(n\pi) = f(x). \tag{17.9}$$

If $f(x)$ for $0 < x < L$ is extended to be an odd periodic function on $(-L, L)$, its sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{17.10}$$

with

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

where we assume that f is sufficiently well behaved, so that its Fourier series converges subject to the Fourier convergence theorem. Comparing coefficients in (17.9) and (17.10), we see that we need $b_n = B_n \sinh(n\pi)$.

In summary, we have proved the following result.

Theorem 17.2 (Solution of a particular Dirichlet problem for the square)**.** If

$$B_n = \frac{2}{L\sinh(n\pi)}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \tag{17.11}$$

then

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi}{L}(L - y)\right) \tag{17.12}$$

satisfies the Dirichlet problem for Laplace's equation in $(0, L) \times (0, L)$ with boundary conditions

$$u(0, y) = u(L, y) = u(x, L) = 0, \quad \text{and} \quad u(x, 0) = f(x). \tag{17.13}$$

Example 17.3. If $f(x) = 1$ in (17.13) then

$$\begin{aligned} B_n &= \frac{2}{L\sinh(n\pi)}\int_0^L \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \frac{2}{n\pi\sinh(n\pi)}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_0^L \\ &= \begin{cases} \dfrac{4}{n\pi\sinh(n\pi)} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}$$

So

$$u(x, y) = \sum_{n \text{ odd}, n\geq 1} \frac{4}{n\pi}\frac{1}{\sinh(n\pi)} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(L - y)}{L}\right). \tag{17.14}$$

17.3 Solution of the Dirichlet problem in a square with four inhomogeneous sides

The solution to Laplace's equation in a square with non-zero Dirichlet conditions on each of the four sides is obtained by superposing four solutions of the type found above; i.e. suppose we want to solve

$$\nabla^2 u(x, y) = 0 \qquad \text{for } (x, y) \in (0, L) \times (0, L)$$

with

$$u(x, 0) = f_1(x), \quad u(0, y) = f_2(y), \quad u(x, L) = f_3(x), \quad u(L, y) = f_4(y). \tag{17.15}$$

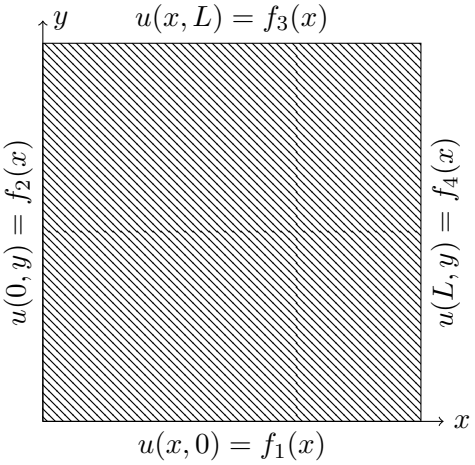


Figure 17.2: The Dirichlet boundary conditions with four inhomogeneous sides.

By the principle of superposition

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

where each of the individual u_i solutions for $i = 1, 2, \dots 4$ satisfies the inhomogeneous condition corresponding to f_i, but homogeneous conditions on the remaining three sides.

The first problem was solved in Theorem 17.2 and the other three can be solved in the same way. [4] Indeed, the separation of variables algorithm yields

$$\begin{aligned}
u_1(x,y) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi\,(L-y)}{L}\right), \\
u_2(x,y) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{L}\right) \sinh\left(\frac{n\pi\,(L-x)}{L}\right), \\
u_3(x,y) &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right), \\
u_4(x,y) &= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{L}\right) \sinh\left(\frac{n\pi x}{L}\right),
\end{aligned}$$

where B_n are defined by (17.11),

$$\begin{aligned}
A_n &= \frac{2}{L\sinh(n\pi)} \int_0^L f_2(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y, \\
C_n &= \frac{2}{L\sinh(n\pi)} \int_0^L f_3(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \\
D_n &= \frac{2}{L\sinh(n\pi)} \int_0^L f_4(y) \sin\left(\frac{n\pi y}{L}\right) \mathrm{d}y.
\end{aligned}$$

[4] Note that splitting the problem into four separate problems can be viewed as following Step 0 of the separation of variables Algorithm 15.2—we zero as many boundary conditions as possible.

Chapter 18

d'Alembert's formula for the wave equation

In this course, we have focused on the study of the heat, wave, and Laplace's equation on bounded domains using the theory of Fourier series. In fact there are many other types of PDEs and problems where such techniques no longer work[1]. Many of these techniques are more sophisticated and will be taught in later courses.

However, the wave equation is atypical as it is possible to write all solutions in a particular form. Consider the one-dimensional wave equation for a twice-differentiable function $u(x,t)$,

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty. \tag{18.1}$$

Theorem 18.1. Every solution of the wave equation (18.1) can be written as a superposition,

$$u(x,t) = F(\xi) + G(\eta) = F(x-ct) + G(x+ct) \tag{18.2}$$

of right- and left-travelling waves. Here, $F(\xi)$ and $G(\eta)$ are arbitrary twice-differentiable functions, each functions of the new variables,

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct. \tag{18.3}$$

Proof. Let us define ξ and η as in (18.3). We seek a solution $u(x,t) = U(\xi,\eta)$. By the transformation of the derivatives:

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial U}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta},$$
$$\frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial U}{\partial \xi} + \frac{\partial \eta}{\partial t}s\frac{\partial U}{\partial \eta} = -c\frac{\partial U}{\partial \xi} + c\frac{\partial U}{\partial \eta}.$$

Note that this can be written in terms of operators as

$$\partial_x u = (\partial_\xi + \partial_\eta)\, U,$$
$$\partial_t u = c\,(-\partial_\xi + \partial_\eta)\, U.$$

It thus follows by the same calculation that,

$$\partial_{xx} u = (\partial_{\xi\xi} + 2\partial_\xi\partial_\eta + \partial_{\eta\eta})\, U,$$
$$\partial_{tt} u = c^2\,(\partial_{\xi\xi} - 2\partial_\xi\partial_\eta + \partial_{\eta\eta})\, U.$$

[1] For example, separation of variables will only work for domains that are sufficiently simple—like rectangles, circles, spheres, etc.

Substitution into the wave equation (18.1) gives

$$c^2\left(\partial_{\xi\xi} - 2\partial_\xi\partial_\eta + \partial_{\eta\eta}\right)U = c^2\left(\partial_{\xi\xi} + 2\partial_\xi\partial_\eta + \partial_{\eta\eta}\right)U.$$

It thus follows that $U_{\eta\xi} = 0$. Hence integrating once gives $U_\eta = q(\eta)$ and integrating again yields

$$U = F(\xi) + G(\eta),$$

and the result follows.

Remark 18.2. The function,

$$F(x - ct) \tag{18.4}$$

represents a wave of constant shape propagating in the positive x-direction with speed c. Similarly, the function

$$G(x + ct) \tag{18.5}$$

represents a wave of constant shape propagating in the negative x-direction with speed c.

Thus, Theorem 18.1 indicates that every solution of the wave equation can be written as a superposition of a left-travelling wave and a right-travelling wave. However, the underlying functions F and G can be so complicated that it becomes very difficult to visually distinguish the two waves. Only in simple situations can the decomposition be seen and intuitively remarked.

When the string occupies the interval $(-\infty, \infty)$, the decomposition of Theorem 18.1 is especially useful as we cannot use separation of variables and Fourier series. However, there is an explicit solution in this case.

Theorem 18.3 (d'Alembert's solution of the wave equation)**.** Consider the wave equation on an infinite string, as modelled by the

$$u_{tt} = c^2 u_{xx}, \tag{18.6a}$$
$$u(0, x) = f(x), \tag{18.6b}$$
$$u_t(0, x) = g(x), \tag{18.6c}$$

for $x \in (-\infty, \infty)$ and $t > 0$. The solution is given by d'Alembert's formula,

$$u(x, t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{18.7}$$

In particular, in the case where the initial velocity is $g(x) \equiv 0$, then

$$u(x, t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big], \tag{18.8}$$

which indicates that the solution is given by the initial displacement, splitting into a right-travelling wave and a left-travelling wave of half the initial height.

Proof. By Theorem 18.1, the solution is given by functions F and G such that

$$u(x, t) = F(x - ct) + G(x + ct). \tag{18.9}$$

We seek to apply the initial conditions. Notice firstly that by the chain rule, the time derivative is

$$u_t(x, t) = -cF'(x - ct) + cG'(x + ct). \tag{18.10}$$

Thus applying the two conditions (18.6b) and (18.6c), we have at $t = 0$, the two equations

$$F(x) + G(x) = f(x), \tag{18.11a}$$
$$-cF'(x) + cG'(x) = g(x). \tag{18.11b}$$

Integrating the second equation gives

$$-F(x)+G(x)=\frac{1}{c}\int_0^x g(s)\,\mathrm{d}s+a, \tag{18.12}$$

where a is constant. Hence combining with the first equation, we have

$$F(x)=\frac{1}{2}\left[f(x)-\frac{1}{c}\int_0^x g(s)\,\mathrm{d}s-a\right], \tag{18.13}$$

$$G(x)=\frac{1}{2}\left[f(x)+\frac{1}{c}\int_0^x g(s)\,\mathrm{d}s+a\right]. \tag{18.14}$$

The solution is given by $F(x-ct)+G(x+ct)$. Thus using the above,

$$u(x,t)=\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s. \tag{18.15}$$

Remark 18.4. In this course, we will only focus on how to interpret d'Alembert's solution visually and in the case where $g(s)=0$, and where $f(x)$ is a sufficiently localised function. See PS10 Q2-3. More involved applications of d'Alembert's solution delves into PDE approach known as the Method of Characteristics.

We will show how d'Alembert's solution is studied visually during the lectures.

18.1 Supplementary: d'Alembert's formula on the finite interval and connection to Fourier series

Although we had derived d'Alembert's formula above for the case of the infinite interval, it is possible to apply the formula as well to the case of finite intervals[**?**]. This is established with the following theorem.

Theorem 18.5 (d'Alembert's formula for the Dirichlet problem on the finite interval)**.** Consider the Dirichlet problem for $t>0$:

$$u_{tt}=c^2u_{xx}\quad \text{for } x\in(0,L), \tag{18.16a}$$

$$u(0,t)=0=u(L,t), \tag{18.16b}$$

$$u(x,0)=u_0(x), \tag{18.16c}$$

$$u_t(x,0)=v_0(x)\equiv 0. \tag{18.16d}$$

Notice that we consider the case where the initial velocity is zero.

Let $\widetilde{u_0}$ be the odd $2L$-periodic extension. The solution of the Dirichlet problem for the wave equation is then given by

$$u(x,t)=\frac{1}{2}\Big(\widetilde{u_0}(x-ct)+\widetilde{u_0}(x+ct)\Big). \tag{18.17}$$

Proof of Theorem 18.5. Using the fact that $v_0=0$ (and hence $B_n=0$) and the trigonometric identity (11.9a), we see that (16.16) becomes

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right),\\&=\frac{1}{2}\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}(x-ct)\right)+\frac{1}{2}\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}(x+ct)\right).\end{aligned} \tag{18.18}$$

Notice that when $y\in(0,L)$, then $\sum_{n=1}^{\infty}A_n\sin n\pi y/L$ equals $u_0(y)$. When $y\in\mathbb{R}\setminus(0,L)$, the same sum equals the odd periodic extension of u_0, i.e. $\widetilde{u_0}$, evaluated at y. Therefore, the right-hand side of (18.18) becomes the right-hand side of (18.17) [where we have $\widetilde{u_0}$ instead of u_0 since $x\pm ct$ might not be in $(0,L)$].

Chapter 19

Uniqueness of PDEs

Throughout the last few chapters, we have implicitly assumed that solutions of the heat, wave, and Laplace's equations are unique. Here we show how solutions to the heat equation or Laplace's equation can be shown to be unique.

19.1 Uniqueness for the 1D heat equation with Dirichlet conditions

Let $u = U(x,t)$ and $u = T(x,t)$ be two solutions of the 1D heat system with zero Dirichlet boundary conditions, as formulated in Definition 14.16:

$$u_t = \kappa u_{xx}, \tag{19.1a}$$
$$u(0,t) = 0 = u(L,t), \tag{19.1b}$$
$$u(x,0) = f(x). \tag{19.1c}$$

Our goal is to show that necessarily $U = T$, and that the solution of the problem is unique. Let $W(x,t) = U(x,t) - T(x,t)$. Then W must satisfy

$$W_t = \kappa W_{xx}, \tag{19.2a}$$
$$W(0,t) = 0 = W(L,t), \tag{19.2b}$$
$$W(x,0) = 0. \tag{19.2c}$$

Notice that, W is the solution of a heat problem for a rod that is held at zero temperature at both ends, and begins at zero temperature throughout. Intuitively, we would expect that $W \equiv 0$.

In order to show this, let

$$E(t) = \frac{1}{2}\int_0^L [W(x,t)]^2 \,\mathrm{d}x. \tag{19.3}$$

This can be considered as a measure of heat energy, where $E(t) \geq 0$. Notice that if we differentiate with respect to time,

$$\begin{aligned} E'(t) &= \int_0^L W W_t \,\mathrm{d}x, \\ &= \int_0^L W(\kappa W_{xx}) \,\mathrm{d}x, \\ &= \kappa \int_0^L \left[\partial_x \left(W W_x\right) - \left(W_x\right)^2\right] \,\mathrm{d}x. \end{aligned} \tag{19.4}$$

The second line follows from substitution of the heat equation. The third line follows from elementary manipulation. The first term can be integrated directly, and then use the boundary conditions of $W = 0$ at $x = 0, L$. That leaves simply

$$E'(t) = -\kappa \int_0^L W_x^2 \, \mathrm{d}x \leq 0. \tag{19.5}$$

This is the crucial result, for it indicates that $E(t)$ (the 'energy') cannot increase. Moreover, we know that at time $t = 0$, the system must begin at $W = 0$. Hence we have the fact that $E(t)$ must begin at $E(0) = 0$ and cannot increase. Therefore $E(t) = 0$ for all time, and

$$E(t) = \int_0^L [W(x,t)]^2 \, \mathrm{d}x = 0. \tag{19.6}$$

The only squared function that integrates to zero on $[0, L]$ is the trivial function. So $W \equiv 0$ and $U = T$.

We have thus proved the following theorem.

Theorem 19.1. The solution of the 1D heat equation with zero Dirichlet conditions in (19.1) is unique.

Remark 19.2. It is possible to derive the equation for $E(t)$ in (19.3) using a more 'organic' way. Begin with the heat equation and multiply by u and integrate over the domain:

$$\int_0^L u u_t \, \mathrm{d}x = \kappa \int_0^L u u_{xx} \, \mathrm{d}x. \tag{19.7}$$

Then write $uu_t = \frac{1}{t}\partial_t u^2$ and use the fact that $\partial_x(uu_x) = u_x^2 + uu_{xx}$ to re-write the right hand-side. This yields exactly (19.4).

19.2 Uniqueness of 2D Laplace's equation with Dirichlet conditions

We wish to prove that solutions to Laplace's equation in Definition 17.1 are unique. Thus consider:

$$\nabla^2 u = 0 \quad \text{for} \quad (x,y) \in \Omega, \tag{19.8a}$$

$$u(x,y) = f(x,y) \quad \text{for} \quad (x,y) \in \partial\Omega. \tag{19.8b}$$

Consider Green's first identity (eqn (1) from PS4). For arbitrary functions u and v in Ω (sufficiently nice so that the divergence theorem applies)

$$\iiint_\Omega \left(u\nabla^2 v + \nabla u \cdot \nabla v\right) \, \mathrm{d}V = \iint_{\partial\Omega} u \frac{\partial v}{\partial n} \, \mathrm{d}S, \tag{19.9}$$

where recall that

$$\frac{\partial v}{\partial n} = \nabla v \cdot \boldsymbol{n}, \tag{19.10}$$

is the directional derivative in the normal direction.

Next, suppose that we have two solutions, say $U(x,y)$ and $V(x,y)$ of the Laplace problem (19.8). Let $W(x,y) = U(x,y) - V(x,y)$. Then we have

$$\nabla^2 W = 0 \quad \text{for} \quad (x,y) \in \Omega, \tag{19.11a}$$

$$W(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega. \tag{19.11b}$$

and choose in (19.9) $u = W$ and $v = W$. Then we have

$$\iiint_\Omega \nabla W \cdot \nabla W \, \mathrm{d}V = \iint_{\partial\Omega} W \frac{\partial W}{\partial n} \, \mathrm{d}S. \tag{19.12}$$

For the right hand-side, $W = 0$ everywhere on the surface. Thus

$$\iiint_{\Omega} (W_x^2 + W_y^2)\,\mathrm{d}V = 0. \tag{19.13}$$

This is only possible if $W_x = 0$ and $W_y = 0$. Hence W is constant. Since $W = 0$ on $\partial\Omega$, this constant must be zero and the solution is unique.

Theorem 19.3. The solution of the 2D Laplace's equation with Dirichlet conditions in (19.8) is unique.

Appendix A

Vector index notation

There is a very powerful and convenient shorthand that we can introduce to help deal with the necessary algebra known as index notation.

Let us write the Cartiesian unit vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ as $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, and vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$. Then notice that we have for the vector $\boldsymbol{a}$,

$$\boldsymbol{a} = \sum_{i=1}^{3} a_i \boldsymbol{e}_i.$$

Rather than explicitly writing the summation sign each time, we shall occasionally suppress the symbol and write instead.

$$\boldsymbol{a} = a_i \boldsymbol{e}_i.$$

Thus the convention of index notation is as follows: whenever the reader sees a repeated index, this is to be interpreted as a summation over that index. In most cases, the lower and upper values of the index (of the summation) should be obvious.

For example, the gradient is now written as

$$\nabla f = \frac{\partial f_i}{\partial x_i} \boldsymbol{e}_i.$$

The power of index notation appears when we introduce shorthand for the dot and cross products. We first define the Kronecker delta as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{A.1}$$

In other words, δ_{ij} is only non-zero if the two indices match.

Now the dot product can be written as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_{ij} b_{ij} \delta_{ij}. \tag{A.2}$$

The reader can verify that this returns $a_1 b_1 + a_2 b_2 + a_3 b_3$ as desired. However, it can also be written more simply as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i. \tag{A.3}$$

We shall also introduce the Levi-Cevita tensor ϵ_{ijk}, defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ cyclic} \\ -1 & \text{if } ijk \text{ acyclic} \\ 0 & = \text{otherwise.} \end{cases} \tag{A.4}$$

Here, by 'cyclic' we mean $ijk = \{123, 312, 231\}$, or that the numbers read from left-to-right in increasing order. Similarly, by 'acyclic', we mean $ijk = \{132, 213, 321\}$, or that numbers read right-to-left in increasing order. Finally, the 'otherwise' condition applies if there is a repeated digit.

The cross product can then be written as

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k. \tag{A.5}$$

For example, the first component is

$$(\boldsymbol{a} \times \boldsymbol{b})_1 = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{1jk} a_j b_k = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2.$$

Notice that only $(j, k) = \{(2, 3), (3, 2)\}$ needs to be handled because any term with a repeated index is zero.

The last useful result to know is what happens if two ϵ_{ijk} tensors are stacked differing by two indices. This is given without proof as[1]

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}. \tag{A.6}$$

Notice that k disappears as it is a summation over k.

It is time to show the power of index notation.

Example A.1 (Triple Scalar Product)**.** Using index notation to prove the triple scalar product

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}), \tag{A.7}$$

for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$.

Solution. We have for the left hand-side,

$$a_i (\boldsymbol{b} \times \boldsymbol{c})_i = a_i (\epsilon_{ijk} b_j c_k)$$

However, since $\epsilon_{ijk} = \epsilon_{jki}$ by a cyclic shift, then the above is re-arranged to

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = b_j (\epsilon_{jki} c_k a_i) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}).$$

The last relation in (A.7) is proved, similarly.

Example A.2 (Triple Vector Product)**.** Using index notation to prove the triple vector product,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}), \tag{A.8}$$

for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$.

[1] A useful mnemonic is "IL-IM" and indicates the first δ factors of the two terms on the right hand-side. The remaining indices follow from elimination.

Appendix B

Useful formulae

	Notation		Calculation	Comments
Line integral (scalar)	$\int_C \phi \, \mathrm{d}s$	$=$	$\int_a^b \phi(\boldsymbol{r}(t)) \, \lvert \boldsymbol{r}'(t) \rvert \, \mathrm{d}t$	
Work line integral	$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$	$=$	$\int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \, \mathrm{d}t$	
Surface integral	$\iint_S \phi \, \mathrm{d}S$	$=$	$\iint_{S_{uv}} \phi(\boldsymbol{r}(t)) \left\lvert \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\rvert \mathrm{d}u\mathrm{d}v$	
Flux integral	$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$	$=$	$\iint_{S_{uv}} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) \mathrm{d}u\mathrm{d}v$	
Divergence theorem	$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$	$=$	$\iiint_V \nabla \cdot \boldsymbol{F} \, \mathrm{d}V$	
Stokes' theorem	$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$	$=$	$\iint_S \nabla \times \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$	