MA20223
Problem sets 2019–2020

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Problem sets for vector calculus

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Problem set 1: Line integrals and conservative fields

PART A

1. Consider the top half of the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $y \geq 1$. The ellipse has mass (per unit area) of $\rho(x, y) = y$.

   (a) Explain using the Riemann sum definition of a double integral why the total mass of the ellipse is given by
   $$M = \iint_S \rho(x, y)\ \mathrm{d}x\,\mathrm{d}y.$$
   Draw a picture of the ellipse, illustrating the notion of the infinitesimal elements of the Riemann sum definition.

   (b) Calculate M directly in Cartesian coordinates in two ways: (i) by integrating in the order of $\mathrm{d}y\,\mathrm{d}x$; (ii) by integrating in the order $\mathrm{d}x\,\mathrm{d}y$.

   (c) Use the transformation $x(r, \theta) = ar\cos\theta$ and $y(r, \theta) = br\sin\theta$ to calculate
   $$M = \iint_{S_{r\theta}} \rho(x(r, \theta),\ y(r, \theta))\ J\ \mathrm{d}r\,\mathrm{d}\theta,$$
   where the Jacobian is given by
   $$J = \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right|.$$
   Make a sketch of the region $S_{r\theta}$ in the (r, θ) plane.

2. (a) Calculate $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ from $(1, 0, 0)^T$ to $(1, 0, 4)^T$ where $\boldsymbol{F}(\boldsymbol{x}) = x\boldsymbol{i} - y\boldsymbol{j} + z\boldsymbol{k}$ and C is

   (i) the helix $\boldsymbol{r}(t) = (\cos 2\pi t, \sin 2\pi t, 4t)$ with $t \in [0, 1]$.

   (ii) the straight line joining the two points.

   Comment on the two results.

   (b) By finding a potential for $\boldsymbol{F}$, obtain your results in (a) from the Fundamental Theorem of Calculus for work integrals.

3. Prove that for a scalar field f and vector field $\boldsymbol{F}$, that
$$\int_{-C} f\,\mathrm{d}s = \int_C f\,\mathrm{d}s \quad \text{and} \quad \int_{-C} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}.$$
Briefly, explain why the difference between the two results.

4. (a) Let
$$\boldsymbol{F}(\boldsymbol{x}) := \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right) \quad \text{for} \quad \boldsymbol{x} \neq 0.$$
By finding an antiderivative, show that there exists some ϕ for which $\boldsymbol{F} = \nabla\phi$.

(b) Let C be the closed circle of radius a centred about the origin, traversed in the counterclockwised sense. Show that

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 2\pi.$$

(c) The above seems to contradict the result that for a conservative field,

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \oint_C \nabla\phi \cdot \mathrm{d}\boldsymbol{r} = 0.$$

Explain why this 'contradiction' is not true.

PART B

5. The Big Theorem indicates equivalency that [1] $\boldsymbol{F}$ is conservative; [2] $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$ for closed contours; and [3] $\int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ for contours that start and end-points.
   (a) Prove 2 $\Rightarrow$ 3 in the Big Theorem on conservative forces.
   (b) Prove 3 $\Rightarrow$ 1 in the Big Theorem on conservative forces.

6. Calculate

$$\int_C (3x^2 + 3y^2)^{1/2}\, \mathrm{d}s,$$

where C is the part of the hyperbola $x^2 - y^2 = 1$ from $(1, 0)$ to $(\cosh 2, \sinh 2)$.

7. Consider the triangular contour, C, that consists of straight line segments joining points $A : (0, 0)$, $B : (2, 0)$, and $C : (2, 1)$. Let $\boldsymbol{F} = (2x + y^2)\boldsymbol{i} + (3y - 4x)\boldsymbol{j}$.
   (a) Evaluate

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

   by integrating in the order of (i) ABC; and (ii) CBA.
   (b) By directly seeking antiderivatives, show that there does not exist any suitably differentiable ϕ (defined anywhere), where $\boldsymbol{F} = \nabla\phi$.

Problem set 2: Surfaces and surface integrals

PART A

1. Consider the surfaces defined by:
   (a) $F(\boldsymbol{x}) = 0$ where $F(\boldsymbol{x}) := \boldsymbol{a} \cdot \boldsymbol{x} - c$ for $\boldsymbol{a} = (a_1, a_2, a_3)$ where $a_3 \neq 0$.
   (b) $\boldsymbol{r} = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$ for $0 \leq \theta < 2\pi$, $0 \leq \phi < \pi$.
   (c) The face ABC for a tetrahedron with vertices at
   $$O : (0,0,0), \quad A : (a,0,0), \quad B : (0,b,0), \quad C : (0,0,c).$$
   (d) $\boldsymbol{r} = (r\cos\theta, r\sin\theta, r\sqrt{3})$ for $0 \leq \theta < 2\pi$, $0 \leq r < \infty$.

   For each part, sketch the surface, and then find the unit normal vector to the surface (there may be multiple ways to derive a normal). For (b)–(d) ensure the normal points outwards.

2. Prove that if S is given by the explicit parametrisation $z = f(x, y)$ then
   $$\mathrm{d}S = \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}x\, \mathrm{d}y.$$
   Show that another way of expressing $\mathrm{d}S$ is
   $$\mathrm{d}S = \frac{\mathrm{d}x\, \mathrm{d}y}{|\widehat{\boldsymbol{n}} \cdot \boldsymbol{k}|},$$
   where $\widehat{\boldsymbol{n}}$ is a unit normal to the surface S.

3. Verify that the surface elements for Q1(b–d) are given by
   $$\mathrm{d}S = a^2 \sin\phi\, \mathrm{d}\theta\, \mathrm{d}\phi, \quad \mathrm{d}S = \sqrt{(c/a)^2 + (c/b)^2 + 1}\ \mathrm{d}x\, \mathrm{d}y, \quad \mathrm{d}S = 2r\, \mathrm{d}r\, \mathrm{d}\theta.$$

4. For the surface in Q1(c) with $a = 3$, $b = 3$, and $c = 6$, show that
   $$\iint_S \boldsymbol{F} \cdot \mathrm{d}\mathbf{S} = 36 \quad \text{when} \quad \boldsymbol{F} = \begin{pmatrix} z \\ 2xz \\ -2y \end{pmatrix}$$
   and $\boldsymbol{n}$ is the unit normal vector to S that points *away* from the origin.

PART B

5. Let S be the hemisphere $\{\boldsymbol{x} : x^2 + y^2 + z^2 = a^2,\ z \geq 0\}$. Write S in spherical polar coordinates.

   (a) Show by computation that

   $$\iint_S \mathrm{d}\mathbf{S} := \iint_S \boldsymbol{n}\,\mathrm{d}S = \pi a^2 \boldsymbol{k}, \tag{1}$$

   where $\boldsymbol{n}$ is the unit normal vector to S that points *away* from the origin.

   (b) In class it was shown that

   $$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\,\mathrm{d}S$$

   can be interpreted as the volume of fluid that passes through S in a unit time. By considering $\boldsymbol{F} = \boldsymbol{e}_i$ for $i = 1, 2, 3$ (the unit vectors), explain why the answer (1) is physically sensible.

6. Let $\boldsymbol{F}$ be a vector field, and let S be the surface of a cuboid with a vertex at $\boldsymbol{x}_0 := (x_0, y_0, z_0)$ and sides of length $\delta x, \delta y, \delta z$ parallel to the coordinate axes. Let $Q := \int_S \boldsymbol{F} \cdot \mathrm{d}\mathbf{S}$. By considering each side of the cube separately and using Taylor's theorem, show that

   $$Q \approx \left(\frac{\partial F_1}{\partial x}(\boldsymbol{x}_0) + \frac{\partial F_2}{\partial y}(\boldsymbol{x}_0) + \frac{\partial F_3}{\partial z}(\boldsymbol{x}_0) \right) \delta x\,\delta y\,\delta z$$

   for $\delta x, \delta y$, and δz small.

Problem set 3: Divergence, curl, and the Divergence Theorem

PART A

1. Find the divergence and curl of

$$\text{(i)} \quad \boldsymbol{F} = \begin{pmatrix} x^2y \\ xy^2 \\ xyz \end{pmatrix}, \qquad \text{(ii)} \quad \boldsymbol{F} = \begin{pmatrix} x\sin y \\ \cos y \\ xy \end{pmatrix}.$$

2. Prove the following for scalar field ϕ, and vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$:
   (a) $\nabla \cdot (\phi \boldsymbol{F}) = (\nabla\phi) \cdot \boldsymbol{F} + \phi(\nabla \cdot \boldsymbol{F})$.
   (b) $\nabla \times (\phi \boldsymbol{F}) = (\nabla\phi) \times \boldsymbol{F} + \phi(\nabla \times \boldsymbol{F})$.
   (c) If ϕ is any solution of Laplace's equation, then $\nabla\phi$ is both solenoidal and irrotational.

   Note that it is typically fastest to use index notation.

3. Define cylindrical coordinates using

$$(x,\, y,\, z) = (\rho\cos\theta, \rho\sin\theta, z),$$

   for $\rho > 0$, $0 \leq \theta < 2\pi$, and $-\infty < z < \infty$. You can use without proof that the volume element is given by

$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z.$$

   (a) By an explicit calculation, show that the surface $\rho = a$ has unit normal $\boldsymbol{n}$ and surface element $\mathrm{d}S$ given by

$$\boldsymbol{n} = \frac{(x, y, 0)}{\sqrt{x^2+y^2}} = (\cos\theta, \sin\theta, 0) \qquad \mathrm{d}S = a\,\mathrm{d}\theta\,\mathrm{d}z.$$

   (b) Let $\boldsymbol{F} = (4x, -2y^2, z^2)$ and let Ω be the region bounded by the curves $x^2+y^2 = 4$, $z = 0$, and $z = 3$. Verify the Divergence Theorem,

$$\iiint\limits_{\Omega} \nabla \cdot \boldsymbol{F}\,\mathrm{d}V = \iint\limits_{\partial\Omega} \boldsymbol{F} \cdot \mathrm{d}\mathbf{S}.$$

   holds by directly computing both sides.

PART B

4. Prove that $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$.

5. Let $\boldsymbol{c}$ be a constant vector (i.e. independent of $\boldsymbol{x}$). Find

$$\nabla \cdot (\boldsymbol{c} \times \boldsymbol{x}) \quad \text{and} \quad \nabla \times (\boldsymbol{c} \times \boldsymbol{x}).$$

6. Verify the divergence theorem for the unit cube $R = [0,1]^3$ with

$$\boldsymbol{F} = \Big((x-1)x^2y,\ (y-1)^2xy,\ z^2-1\Big).$$

Problem set 4: Green's theorem and Stokes' theorem

PART A

1. Let $\boldsymbol{F} = (x^2y, xy^2)$. Let C be the closed contour defined by the straight line, C_1, joining $(1, \mp 2)$; the arc of the parabola $y^2 = 4x$, C_2, between the points $(1, \pm 2)$. Orient C in the anticlockwise sense and let Ω be the region enclosed by C.
   (a) Choose a parameterization for C_1 and C_2 using position vectors $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$.
   (b) Calculate
   $$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathrm{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{\mathrm{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$
   (c) Verify that
   $$\iint_\Omega \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A = \frac{104}{105},$$
   and hence verify that Green's theorem in the plane holds for this Ω, C, and $\boldsymbol{F}$.

2. (Stokes' theorem) Let S be the planar surface through points $\{(a, 0, 0), (0, b, 0), (0, 0, c)\}$. Let C be the boundary curve of S.
   (a) With the outward normal, $\boldsymbol{n}$, defined as in PS2, draw a picture of S and show the orientation of C necessary for S and C to be correspondingly oriented.
   (b) As in PS2 Q2, use $a = 3$, $b = 3$, $c = 6$. Let $\widetilde{\boldsymbol{F}} = (xz^2, -2xy, yz)$. Find $\nabla \times \widetilde{\boldsymbol{F}}$ and hence find $\oint_C \widetilde{\boldsymbol{F}} \cdot \mathrm{d}\boldsymbol{r}$.

3. (Integration by parts and Green's identities)
   (a) By using the fundamental theorem of calculus and the product rule, show that
   $$\int_a^b \phi \frac{\mathrm{d}\psi}{\mathrm{d}x} \, \mathrm{d}x = \left[\phi\psi\right]_a^b - \int_a^b \psi \frac{\mathrm{d}\phi}{\mathrm{d}x} \mathrm{d}x.$$
   (b) Let Ω, $\partial\Omega$, and $\boldsymbol{n}$ be as in the statement of the divergence theorem. Let ϕ be a scalar field and $\boldsymbol{F}$ a vector field. By using the divergence theorem, show that
   $$\iiint_\Omega \phi \, \nabla \cdot \boldsymbol{F} \, \mathrm{d}V = \iint_{\partial\Omega} \phi \, \boldsymbol{F} \cdot \mathrm{d}\mathbf{S} - \iiint_\Omega \boldsymbol{F} \cdot (\nabla\phi) \, \mathrm{d}V.$$
   What is the connection of this to (a)?
   (c) Let Ω, $\partial\Omega$, and $\boldsymbol{n}$ be as in the statement of the divergence theorem, and let u and v be two (sufficiently nice) scalar fields. Define the normal derivative of v, $\partial v/\partial n$, by
   $$\frac{\partial v}{\partial n}(\boldsymbol{x}) \equiv \boldsymbol{n}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega,$$
   and similarly for $\partial u/\partial n$. Prove Green's first identity:
   $$\iiint_\Omega u \nabla^2 v \, \mathrm{d}V = \iint_{\partial\Omega} u \frac{\partial v}{\partial n} \, \mathrm{d}S - \iiint_\Omega \nabla u \cdot \nabla v \, \mathrm{d}V,$$
   From this, conclude Green's second identity:
   $$\iiint_\Omega \left(u\nabla^2 v - v\nabla^2 u\right) \, \mathrm{d}V = \iint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S.$$

PART B

4. Using Green's theorem, prove that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint x\,\mathrm{d}y - y\,\mathrm{d}x.$$

   Use this to find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

5. Use Green's theorem in the plane to calculate $\int_C \boldsymbol{F}\cdot\mathrm{d}\mathbf{r}$ where $\boldsymbol{F} = (x\cos y, x^2\sin y)$ and C is the boundary of the region $\{(x, y) : 1 + x^2 \leq y \leq 2, x \geq 0\}$.

6. Verify Stokes' theorem for $\boldsymbol{F} = (y - z + 2, yz + 4, -xz)$,

$$\iint_S (\nabla\times\boldsymbol{F})\,\mathrm{d}\mathbf{S} = \oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r},$$

   where the surface, S, is the surface of the cube $\Omega = [0, 2]^3$ lying above the xy-plane, open along $z = 0$. That is, the face $\{(x, y) : z = 0, 0 \leq x \leq 2, 0 \leq y \leq 2\}$ is removed from the cube.

Problem sets for PDEs

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Problem set 5: Introduction to PDEs

PART A

1. Let $T(\boldsymbol{x}, t) = T(x, y, z, t)$ be the temperature at any point, $\boldsymbol{x} = (x, y, z)$ and time, t, in a closed volume, $V \subseteq \mathbb{R}^3$. Let the density, specific heat, and thermal conductivity be constant and written as ρ, c, and κ, respectively. We wish to derive the three-dimensional heat equation
$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right),$$
where $\kappa = k/(\rho c)$.

   (a) Let $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x}, t)$ be the heat flux vector (positive for flux of heat leaving the solid). Explain why energy conservation implies that
$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho c T \, \mathrm{d}V = - \iint_S \boldsymbol{q} \cdot \mathrm{dS},$$
   where $S = \partial V$ is the bounding surface for V.

   (b) By substituting Fourier's Law, $\boldsymbol{q} = -k \nabla T$, and applying the divergence theorem to the right hand-side, derive the heat equation above. You may assume that the time derivative can be passed through the integral.

2. Derive the wave equation
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$
for a transverse displacement $y(x, t)$ of a stretched string, and where $c = \sqrt{T/\rho}$ for tension T and density ρ.

3. Consider the heat equation $u_t = \kappa u_{xx}$ for a 1D rod $x \in [0, L]$.

   (a) What do the two types of boundary conditions
$$\begin{cases} u(0,t) = T_0, \\ u(L,t) = T_1, \end{cases} \qquad \text{and} \qquad \begin{cases} u_x(0,t) = F_0, \\ u_x(L,t) = F_1, \end{cases}$$
   mean in terms of physical quantities of temperature and heat flux (or heat flow)? Assume $\{T_0, T_1, F_0, F_1\}$ are all constant.

   (b) What does it mean to say that the boundary at $x = L$ is thermally insulated?

   (c) Consider now the heat equation defined on a 3D volume V as in Q1. How do you generalise the two types of boundary conditions? What does it mean for surface to be thermally insulated?

4. Consider the wave equation for $y = y(x,t)$ with fixed end conditions:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$
$$y(0,t) = 0 = y(L,t), \quad t > 0.$$

By separation of variables and writing $y(x,t) = X(x)T(t)$, show that the general solution can be written as a superposition of modes,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right],$$

for constants a_n and b_n.

Problem set 6: Fourier series I

PART A

1. Which of the following functions are periodic, even, odd or neither about $x = 0$? Of those that are periodic, find the fundamental period (i.e. smallest a such that $f(x + a) = f(x)$).
   (a) $f(x) = \sin(x)$,
   (b) $f(x) = \sin^2(x)$,
   (c) $f(x) = \sin(x)\cos(3x)$,
   (d) $f(x) = \sin(x) + \sin(\sqrt{2}x)$,
   (e) $f(x) = \sin(x)e^{-\cos(x^2)}$,
   (f) $f(x) = 1, \quad x \in (-1, 0], \quad f(x) = 2 - x \quad x \in (0, 1], \quad f(x) = f(x + 2)$.

2. Let $f(x)$ be the 2π-periodic function defined by Let $f(x)$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} -\dfrac{\pi}{2} - \dfrac{x}{2} & x \in (-\pi, 0) \\ \dfrac{\pi}{2} - \dfrac{x}{2} & x \in (0, \pi) \end{cases}$$

   and $f(x) = f(x + 2\pi)$.
   (a) Sketch the function $f(x)$, and in particular the interval $(-\pi, \pi]$.
   (b) Show that $f(x)$ has the Fourier series given by

$$f(x) \sim \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

   (c) By assuming that the above infinite sum converges to $f(\pi/2)$ at $x = \pi/2$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots.$$

3. Let $f(x)$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} -x & x \in (-\pi, 0) \\ x & x \in (0, \pi) \end{cases}$$

   and $f(x) = f(x + 2\pi)$.
   (a) Sketch the function $f(x)$.
   (b) Show directly that $f(x)$ has the Fourier series given by

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos[(2k + 1)x]}{(2k + 1)^2}.$$

   (c) Assuming that the Fourier series converges to $f(x)$ for all x, evaluate $f(x)$ at a suitable point to show that

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^2}.$$

$\boxed{\textbf{PART B}}$

4. This follows a similar derivation of the notes, but for Fourier series about $[-L, L]$ instead of $[-\pi, \pi]$. Use this opportunity to better understand the manipulations that were used to derive the Fourier coefficients.

   Let us call

$$C_n = \cos\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad S_n = \sin\left(\frac{n\pi x}{L}\right).$$

   (a) Show that, for $m, n > 0$,

$$\begin{aligned}
\langle C_n,\, C_m\rangle &= \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = L\delta_{mn}\\
\langle S_n,\, C_m\rangle &= \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = 0\\
\langle S_n,\, S_m\rangle &= \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = L\delta_{mn}
\end{aligned}$$

   where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ if $m \neq n$.

   For the special case of $n = 0 = m$, show that

$$\begin{aligned}
\langle C_0,\, C_0\rangle &= 2L\\
\langle S_0,\, C_0\rangle &= 0\\
\langle S_0,\, S_0\rangle &= 0
\end{aligned}$$

   (b) Consider a piecewise differentiable function $f(x)$ on $[-L, L]$. Consider the Fourier series representation

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right].$$

   Derive formulae for A_n and B_n.

Problem set 7: Fourier series II

PART A

1. (a) In lectures, we worked out the Fourier series to the 2π periodic extension of $f(x) = e^x$ for $x \in (0, 2\pi)$. By evaluating the Fourier sum at $x = 0$, show that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2}\left(\frac{e^{2\pi}+1}{e^{2\pi}-1}\right) - \frac{1}{2}.$$

   (b) In lectures, we worked out the Fourier series to the 2π even periodic extension of $f(x) = e^x$ for $x \in (0, \pi)$. By evaluating the Fourier sum at $x = 0$, show that

$$1 = \frac{1}{\pi}(e^{\pi} - 1) + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi} - 1}{1+n^2}.$$

2. (a) Define the 2π-periodic function from $f(x) = x\sin(px)$ for $x \in (-\pi, \pi)$, where p is a positive integer. Sketch the function. Is it even or odd?

   (b) Calculate the Fourier series of f.

   (c) By considering values of the series at $x = 0$ or $x = \pi$ for $p = 1$ or $p = 2$, calculate

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \quad \text{and} \quad \sum_{n=3}^{\infty} \frac{1}{n^2-4}.$$

3. Let $f(x) = x^2, \quad x \in (0, 1)$.

   (a) Show that if $f(x)$ is extended to be a periodic function of period $2L = 1$, then this has the Fourier Series

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{(n\pi)^2}\cos(2n\pi x) - \frac{1}{n\pi}\sin(2n\pi x)\right),$$

   Sketch $f(x)$.

   (b) By extending $f(x)$ to be an *odd* periodic function of period $2L = 2$, show that the Fourier Sine series for $f(x)$ takes the form

$$f(x) \sim \sum_{n=1}^{\infty}\left(-\frac{2(-1)^n}{n\pi} + \frac{4}{(n\pi)^3}\Big((-1)^n - 1\Big)\right)\sin(n\pi x),$$

   Sketch $f(x)$.

   (c) By extending $f(x)$ to be an *even* periodic function of period $2L = 2$, find the Fourier cosine series for $f(x)$ and sketch the resulting function.

   (d) Which of the series in parts (b) and (c) do you expect to converge faster? Why?

$\boxed{\text{PART B}}$

4. Consider the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

which can be used to represent 2π-periodic functions.

(a) Using the standard derivation of the Fourier coefficients, derive the analogous expressions

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x,$$
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x, \quad n \neq 0.$$

(b) Let $f(x) = e^x$ for $x \in (-\pi, \pi)$ with Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

with $|x| > \pi$. Differentiating gives us

$$e^x = \sum_{n=-\infty}^{\infty} (in)c_n e^{inx},$$

so using the Fourier series on the left gives

$$c_n = (in)c_n,$$

for all n. Thus $c_n = 0$.

Where is the mistake?

Problem set 8: The heat equation

PART A

1. The **homogeneous Neumann problem** for the 1D heat equation is given by

$$u_t = u_{xx} \quad \text{on } x \in [0, L],$$
$$u_x(0,t) = 0 = u_x(L,t), \quad \text{and} \quad u(x,0) = f(x).$$

   (a) By applying separation of variables to the above system, show that the family of separable solutions to the BVP is given by

$$u_n(x,t) = A_n \exp\left(-\frac{n^2\pi^2}{L^2}t\right)\cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \ldots$$

   (b) Suppose that $f(x)$ is specified on $[0, L]$ and construct the even extension of f to $[-L, L]$. Using the principle of superposition, find an expression for the solution to the homogeneous Neumann problem. Your solution should involve the Fourier cosine coefficients,

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad n = 0, 1, 2, \ldots$$

2. Find an expression for the solution of the **inhomogeneous Dirichlet problem** for the heat equation on $[0, L]$ (with $\kappa = 1$) when

   (a) $u(0,t) = 0$, $u(L,t) = 1$, $u(x,0) = 0$.

   (b) $u(0,t) = 0$, $u(L,t) = 1$, $u(x,0) = \sin(\pi x/L)$.

   In both cases, sketch $u(x,t)$ against x for several values of t showing the approximate qualitative behaviour.

3. In the separation of variables procedure for the **homogeneous Dirichlet problem** with zero Dirichlet BCs at both ends,

$$u(0,t) = 0 = u(L,t),$$

   we use the fact that (for $\kappa = 1$)

$$\frac{G'(t)}{G(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 < 0.$$

   If instead the constant on the right is positive or zero, $\lambda^2 \geq 0$, show that only the trivial solution is produced once the boundary conditions are applied.

   What happens if we have **homogeneous Neumann conditions**,

$$u_x(0,t) = 0 = u_x(L,t)$$

   at both ends?

PART B

4. Consider the heat equation in 2D, given by $T = T(x, y, t)$ where

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

(a) Consider the steady-state solution of the heat equation in a disc of radius $r \leq a$. By writing $T = T(r, \theta)$ and using Laplace's equation in plane polar coordinates,

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0,$$

show that separable solutions must take the form of

$$T_n(r, \theta) = r^n \Big(A \cos n\theta + B \sin n\theta \Big),$$

for non-negative integer n, and where A and B are constants. Carefully explain why only these solutions are possible.

(b) Find the steady-state temperature in the form of an infinite series when the temperature along the boundary of the disc, $r = a$, is given by

$$T(a, \theta) = \begin{cases} 1 & \text{for } 0 \leq \theta < \pi, \\ 0 & \text{for } \pi \leq \theta < 2\pi. \end{cases}$$

Problem set 9: The wave equation

PART A

1. Consider the **homogeneous Neumann problem** for the 1D wave equation on $x \in [0, L]$:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u_x(0,t) = 0 &= u_x(L,t), \\ u(x,0) &= u_0(x), \\ u_t(x,0) &= v_0(x). \end{aligned} \tag{$\dagger$}$$

   (a) By duplicating the separation of variables procedure, show that the family of separable solutions takes the form,

$$u_0(x,t) = \frac{1}{2}\Big(A_0 + B_0 t\Big),$$

   and

$$u_n(x,t) = \cos\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \quad n = 1, 2, \ldots \tag{$\ddagger$}$$

   (b) Plot the solution ($\ddagger$) with $n = 1$, $A_1 = 1$, $B_1 = 0$ for $t = 0,\ L/c,\ 2L/c$.

   (c) Plot the solution ($\ddagger$) with $n = 2$, $A_2 = 1$, $B_2 = 0$, for $t = 0,\ L/(2c),\ L/c$.

   (d) Apply the initial conditions in the system ($\dagger$) and write down the Fourier series solution.

2. The ends at $x = 0$ and $x = L$ of a flexible string are fixed with zero displacement. The point p with $0 < p < L$ is drawn a distance h and at $t = 0$, the string is released from rest. This corresponds to the initial conditions of

$$u(x,0) = \begin{cases} hx/p & \text{for } x \in [0,p) \\ h(L-x)/(L-p) & \text{for } x \in (p,L] \end{cases} \quad \text{and} \quad u_t(x,0) = 0.$$

   Use separation of variables and Fourier series to find $u(x,t)$.

3. (a) By introducing new independent variables $\xi = x - ct$ and $\eta = x + ct$, show that if $u(x,t)$ is a solution of the wave equation,

$$u_{tt} = c^2 u_{xx},$$

   then there are arbitary functions F and G such that

$$u(x,t) = F(x - ct) + G(x + ct).$$

   (b) Show that if u satisfies the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x), \quad x \in (-\infty, \infty),$$

   then

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

4. (a) At time $t = 0$, the displacement of an infinitely long string is defined as:

$$u(x,0) = \begin{cases} \sin(\pi x/a), & |x| \le a, \\ 0, & |x| > a, \end{cases} \quad \text{and} \quad u_t(x,0) = 0, \quad x \in (-\infty, \infty).$$

Using d'Alembert's solution, and assuming that waves may move along the string with speed c, sketch the displacement of the string at $t = \{0,\ a/(2c),\ a/c,\ 3a/(2c)\}$.

(b) Repeat the above but for the initial displacement replaced by

$$u(x,0) = \begin{cases} \cos[\pi x/(2a)], & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

PART B

5. Return to the separation of variables argument for the wave equation, $u_{tt} = c^2 u_{xx}$. Setting $u(x,t) = X(x)G(t)$, we obtain

$$\frac{G''}{c^2 G} = \frac{X''}{X} = C = \text{constant}.$$

In the two cases where $C > 0$ and $C = 0$, and for zero Dirichlet conditions,

$$u(0,t) = 0 = u(L,t),$$

show that only the trivial solution results. What happens if instead we have zero Neumann conditions $u_x(0,t) = 0 = u_x(L,t)$?

6. Consider the wave equation $u_{tt} = c^2 u_{xx}$ defined for the vertical displacement, $u(x,t)$, of a 1D string $x \in [0, L]$. Discuss the physical interpretation of the Dirichlet and Neumann conditions in terms of the quantities of displacement and force.

Problem set 10: Energy and uniquness

PART A

In this set, you will show that solutions of Poisson's equation, the heat equation, and the wave equation are unique given certain boundary conditions. Consider the following systems defined on a volume $V \subseteq \mathbb{R}^3$ with boundary $S = \partial V$.

	Poisson's equation	Heat equation	Wave equation
Equation	$\nabla^2 u = f(\boldsymbol{x})$	$u_t = \kappa \nabla^2 u$	$u_{tt} = c^2 \nabla^2 u$
Initial condition	N/A	$u(\boldsymbol{x}, 0) = u_0$	$u(\boldsymbol{x}, 0) = U_0(\boldsymbol{x})$ $u_t(\boldsymbol{x}, 0) = V_0(\boldsymbol{x})$

The domains of the above functions are defined in the usual way, in consideration of $\boldsymbol{x} \in V$ and $t \geq 0$. Three possible boundary conditions, defined on $S = \partial V$, are now given for the above three systems.

Firstly, we may consider **Dirichlet** conditions:

$$u = F(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial V. \tag{D}$$

Or we may consider **Neumann** conditions:

$$\frac{\partial u}{\partial n} = G(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial V. \tag{N}$$

Or we may consider **mixed** conditions:

$$Au + B\frac{\partial u}{\partial n} = H(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial V, \tag{M}$$

where in the above, you may assume that A and B are constant, and A has the same sign as B.

1. For each of the three equations above:
   (a) Begin by setting $w = u - v$ for two arbitrary solutions u and v. State the system of equations that w must satisfy.
   (b) Derive an expression for the energy E (of Poisson's equation) or the evolution of energy $E(t)$ (of the heat/wave equations).
   (c) Prove that solutions are either unique or defined up to a constant for the three boundary conditions of (D), (N), or (M).

   You may use the vector identity

$$\nabla \cdot (w_1 \nabla w_2) = \nabla w_1 \cdot \nabla w_2 + w_1 \nabla^2 w_2.$$

Problem set Z

Problem set Z consists of problems (with examinable content) that are added, if required, as the term progresses.

HINTS FOR VECTOR CALCULUS

Problem set 1

1. The mass is $M = 2ab^2/3$.

2. For (b) find the potential such that $\boldsymbol{F} = \nabla\phi$ and calculate the integral directly.

3. In order to do the computation, note that $-C$ can be defined by a new parameterization $\tilde{r}(t) = r(-t)$ for $t \in [-b, -a]$.

4. For (a) consider $\phi = \tan^{-1}(y/x)$. For (b) use polar coordinates.

5. The proof for (a) is straightforward.

   The proof for (b) is more subtle. Set it up as follows. Given $\boldsymbol{F}$ satisfying

   $$\int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \tag{21}$$

   for two contours C_1 and C_2 sharing the same start and end-points, assume without loss of generality that the start point is 0 and the endpoint is $\boldsymbol{x}$. Let

   $$\phi(\boldsymbol{x}) := \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

   where C is any curve with start point 0 and endpoint $\boldsymbol{x}$. Note that $\phi(\boldsymbol{x})$ is indeed a scalar function of $\boldsymbol{x}$ (since the value of the integral is a real number), and $\phi(\boldsymbol{x})$ is well defined since, by (21), the integral is independent of the path connecting 0 and $\boldsymbol{x}$.

   By letting $C = C_1 \cup C_2$ where C_1 is the straight-line path from $0 = (0, 0, 0)$ to $(0, y, z)$, and C_2 is the straight-line path from $(0, y, z)$ to (x, y, z), show that

   $$\frac{\partial\phi}{\partial x}(\boldsymbol{x}) = F_1(\boldsymbol{x}).$$

   Use analogous arguments to show that

   $$\frac{\partial\phi}{\partial y}(\boldsymbol{x}) = F_2(\boldsymbol{x}) \quad \text{and} \quad \frac{\partial\phi}{\partial z}(\boldsymbol{x}) = F_3(\boldsymbol{x}).$$

6. The endpoints of C give you a hint of how to choose $\boldsymbol{r}$(t).

7. (a) The integral in the direction ABC is $-14/3$. (b) The question is asking you to integrate, for example, $\partial\phi/\partial x = 2x + y^2$. Then differentiate to match with $\partial\phi/\partial y = 3y - 4x$.

Problem set 2

1. You have a number of ways to obtain the normal. For example, given a level set $F(x, y, z) =$ const. the normal can be calculated by $\boldsymbol{n} = \nabla F/|\nabla F|$.

   Alternatively, given a parameterisation $\boldsymbol{r}(u, v)$, the normal can be obtained by a cross-product formula found in the notes.

2. For the first part of the question, write $\boldsymbol{r} = (x, y, f(x, y))$ and use cross-product formula.

3. This follows by direct calculation. Either use the formula for $\mathrm{d}S$ in terms of cross products or use the results from Q2.

4. Use $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S$ and insert the components from previous questions. Calculate $\boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S$ and reduce everything to a function of x and y.

   Once you have done this, draw the (triangular) shape of the region of integration in the (x, y)-plane and carefully determine the bounds of integration. Integrate!

5. (a) Use the fact that the normal vector at a given point on the sphere is the radial vector that joins the origin to the given point; if you have done things right, the first two components of the integral will be zero; (b) draw the vector field.

6. —

Problem set 3

1. Straightforward.

2. Straightforward.

3. (a) Start by writing down $\boldsymbol{r}(\theta, z)$. Calculate $\boldsymbol{n}$ and $\mathrm{d}S$ using cross product formula. (b) The LHS is a standard integration which you should do using cylindrical coordinates; for the RHS, write down

$$\left(\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}}\right) \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

   Write out the normal for each integral, compute the integrand, and finally compute the integral.

4. Index notation saves a great deal of time if you can figure out how it is done!

5. Use the previous question for the first part; the second part can be done manually via each of the components.

6. The function $\boldsymbol{F}$ was cooked up so that the surface normals for each side of the cube are all particularly simple.

Problem set 4

1. The integral over C_1 is $16/3$ and the integral over C_2 is $-152/35$.

2. —

3. (a) Use integration by parts.
   (b) Recall $\nabla \cdot (\phi \boldsymbol{F}) = (\nabla \phi) \cdot \boldsymbol{F} + \phi(\nabla \cdot \boldsymbol{F})$. You want to apply the divergence theorem—what should be the substitution for the vector that appears in the theorem?
   (c) Use the results of part (b); what needs to be used for ϕ and $\boldsymbol{F}$?

4. To calculate the line integral via Green's theorem, use a parameterisation $\boldsymbol{r}(\theta)$ that works for the ellipse.

5. Straightforward.

6. There should be five faces involved. Which face is the open side? For each of the five faces, start by writing out the outwards normal. There should also be four relevant contours that form C. Write out a parameterisation for each.

HINTS FOR PDES

To be written during the course of the term!