# MA40255 Viscous Fluid Dynamics: Lecture Notes

## Contents

# 0 Course synopsis

## 0.1 Overview

Viscous fluids are important in so many facets of everyday life that everyone has some intuition about the diverse flow phenomena that occur in practice. This course is distinctive in that it shows how quite advanced mathematical ideas such as asymptotics and partial differential equation theory can be used to analyse the underlying differential equations and hence give scientific understanding about flows of practical importance, such as air flow round wings, oil flow in a journal bearing and the flow of a large raindrop on a windscreen.

## 0.2 Reading list

[1] D.J. Acheson, *Elementary Fluid Dynamics* (Oxford University Press, 1990), chapters 2, 6, 7, 8. ISBN 0198596790.
[2] H. Ockendon & J.R. Ockendon, *Viscous Flow* (Cambridge Texts in Applied Mathematics, 1995). ISBN 0521458811.

## 0.3 Further reading

[3] G.K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, 2000). ISBN 0521663962.
[4] C.C. Lin & L.A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences* (Society for Industrial and Applied Mathematics, 1998). ISBN 0898712297.
[5] L.A. Segel, *Mathematics Applied to Continuum Mechanics* (Society for Industrial and Applied Mathematics, 2007). ISBN 0898716209.

## 0.4 Synopsis (20 lectures)

Euler's identity and Reynolds' transport theorem. The continuity equation and incompressibility condition. Cauchy's stress theorem and properties of the stress tensor. Cauchy's momentum equation. The incompressible Navier-Stokes equations. Vorticity. Energy. Exact solutions for unidirectional flows; Couette flow, Poiseuille flow, Rayleigh layer, Stokes layer. Dimensional analysis, Reynolds number. Derivation of equations for high and low Reynolds number flows.

Thermal boundary layer on a semi-infinite flat plate. Derivation of Prandtl's boundary-layer equations and similarity solutions for flow past a semi-infinite flat plate. Discussion of separation and application to the theory of flight.

Slow flow past a circular cylinder and a sphere. Non-uniformity of the two dimensional approximation; Oseen's equation. Lubrication theory: bearings, squeeze films, thin films; Hele-Shaw cell and the Saffman-Taylor instability.

# 1 The Navier-Stokes equations

## 1.1 Motivation for studying viscous fluids

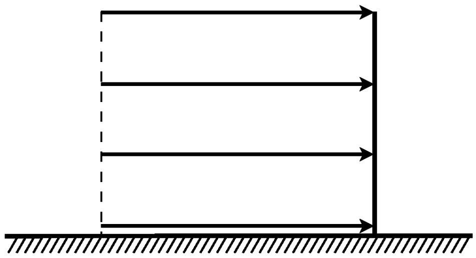
- Fluid mechanics is the study of the flow of liquids and gases.
- In many practical situations the fluid can be described effectively as incompressible and inviscid, and modelled by the Euler equations

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

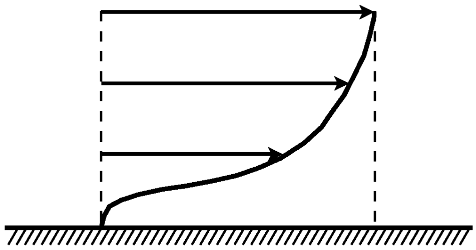
$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \rho \mathbf{F}, \tag{2}$$

  where the velocity $\mathbf{u}$ and pressure $p$ are functions of position $\mathbf{x}$ and time $t$, $\rho$ is the constant density and $\mathbf{F}$ is the external body force acting per unit mass (*e.g.* gravity).

- For example:
  - (i) aerodynamic flows (*e.g.* flow past wings);
  - (ii) free surface flows (*e.g.* water waves).
- However, there are many fluid flow phenomena where inviscid theory fails, *e.g.*
  - (i) D'Alembert's paradox states that there is no drag on an object moving steadily through a fluid (*cf.* a ball bearing falling through oil).
  - (ii) The ability of a thin layer of fluid to support a large pressure, *e.g.* a lubricated bearing.
  - (iii) You can't clean dust from a car by driving fast. Inviscid flow allows slip between car and air:



  The tenacious dust suggests that fluid adjacent to the car is dragged along with it:



- The problem with inviscid fluid mechanics which gives rise to these failings, is that it takes no account of the friction caused by one layer of fluid sliding over another or over a solid object.
- This friction is related to the stickiness, or viscosity, of real fluids.
- No fluid is completely inviscid (except liquid helium below around 1 Kelvin).
- Even for low-viscosity fluids ( *e.g.* air), there will often be regions (*e.g.* thin boundary layer on a moving car) where viscous effects are important.

- In this course we will see how the Euler equations (1)–(2) must be modified to obtain the incompressible Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{F}, \tag{4}$$

for flows in which the viscosity $\mu$ is important.

- The incompressible Navier-Stokes equations (3)-(4) are nonlinear and in general extremely difficult to solve.

- We will use a combination of asymptotics and partial differential equation theory to analyse (3)–(4) and hence give scientific understanding about flows of practical importance.

## 1.2 The summation convention and revision of vector calculus

- We will work in Cartesian coordinates $Oxyz$ and let

$$\mathbf{i} = (1,\ 0,\ 0) = \mathbf{e}_1,\ \mathbf{j} = (0,\ 1, 0) = \mathbf{e}_2,\ \mathbf{k} = (0,\ 0,\ 1) = \mathbf{e}_3,$$

denote the standard orthonormal basis vectors, so that a position vector may be written

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

where $(x_1,\ x_2,\ x_3)$ are the Cartesian coordinates.

- We employ the summation convention of summing over all possible repeated indices in an expression.

- An index which is summed in this way is called a dummy index.

- The summation convention should only be used if it is clear from the context over what ranges the dummy indices should be summed.

- *Example 1*: Denote by $(u,\ v,\ w) = (u_1,\ u_2,\ u_3)$ the components of the fluid velocity, so that

$$\mathbf{u} = (u,\ v,\ w) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \sum_{i=1}^{3} u_i\mathbf{e}_i = u_i\mathbf{e}_i.$$

- *Example 2*: Kronecker's delta $\delta_{ij}$ is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

while the Levi-Civita symbol $\varepsilon_{ijk}$ is defined by

$$\varepsilon_{ijk} = \mathbf{e}_i \cdot (\mathbf{e}_j \wedge \mathbf{e}_k) = \begin{cases} 1 & \text{if } i,\ j,\ k \ \text{ in cyclic order}, \\ -1 & \text{if } i,\ j,\ k \ \text{ in acyclic order}, \\ 0 & \text{otherwise} \end{cases}$$

The alternating tensor $\epsilon_{ijk}$:

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$$
$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$$

In the identity

$$\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr},$$

the sum is over $k = 1, 2, 3$.

- *Example 3*: The determinant of a matrix $A = \{a_{ij}\}_{3\times 3}$ is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}.$$

- *Example 4*: The scalar and vector product of two vectors $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_i \mathbf{e}_i$ are given by

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i, \\ \mathbf{a}\wedge\mathbf{b} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} \mathbf{e}_i a_j b_k. \end{aligned}$$

- *Example 5*: For a differentiable scalar field $f(\mathbf{x})$ and a differentiable vector field $\mathbf{G}(\mathbf{x}) = G_i(\mathbf{x})\mathbf{e}_i$,

$$\boldsymbol{\nabla} f = \mathbf{e}_i \frac{\partial f}{\partial x_i}, \qquad \boldsymbol{\nabla}\cdot\mathbf{G} = \frac{\partial G_j}{\partial x_j}, \qquad \boldsymbol{\nabla}\wedge\mathbf{G} = \varepsilon_{ijk}\mathbf{e}_i \frac{\partial G_k}{\partial x_j},$$

$$(\mathbf{u}\cdot\boldsymbol{\nabla}) f = u_l \frac{\partial f}{\partial x_l}, \qquad \boldsymbol{\nabla}^2 f = \frac{\partial^2 f}{\partial x_m \partial x_m}.$$

Note: $\nabla\wedge\mathbf{G} = \epsilon_{ijk}\mathbf{e}_i \frac{\partial}{\partial x_j} G_k = \epsilon_{ijk}\mathbf{e}_i \frac{\partial G_k}{\partial x_j}$

- *Example 6*: Since $\varepsilon_{ijk} = \mathbf{e}_i \cdot (\mathbf{e}_j \wedge \mathbf{e}_k)$ ,

$$\begin{aligned} \boldsymbol{\nabla}\wedge\mathbf{G} &= (\mathbf{e}_i\cdot(\mathbf{e}_j\wedge\mathbf{e}_k))\mathbf{e}_i \frac{\partial G_k}{\partial x_j} \\ &= \mathbf{e}_j \wedge \mathbf{e}_k \frac{\partial G_k}{\partial x_j} \\ &= \mathbf{e}_j \wedge \frac{\partial}{\partial x_j}(G_k \mathbf{e}_k) \\ &= \mathbf{e}_j \wedge \frac{\partial \mathbf{G}}{\partial x_j}. \end{aligned}$$

- *Example 7*: The identities of vector calculus may be readily derived using these definitions and vector identities. *E.g.* for a differentiable vector field $\mathbf{u}$,

  Using *Example 6:*

$$\begin{aligned} \boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u}) &= \mathbf{e}_i \wedge \frac{\partial}{\partial x_i}\left(\mathbf{e}_j \wedge \frac{\partial \mathbf{u}}{\partial x_j}\right) \\ &= \frac{\partial^2}{\partial x_i \partial x_j}(\mathbf{e}_i \wedge (\mathbf{e}_j \wedge \mathbf{u})) \\ &= \frac{\partial^2}{\partial x_i \partial x_j}((\mathbf{e}_i\cdot\mathbf{u})\mathbf{e}_j - (\mathbf{e}_i\cdot\mathbf{e}_j)\mathbf{u}) \\ &= \frac{\partial^2}{\partial x_i \partial x_j}(u_i \mathbf{e}_j - \delta_{ij}\mathbf{u}) \\ &= \mathbf{e}_j \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right) - \frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_j} \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}^2\mathbf{u}. \end{aligned}$$

*Or* : $$\begin{aligned} \nabla\wedge(\nabla\wedge\mathbf{u}) &= \epsilon_{ijk}\mathbf{e}_i \frac{\partial}{\partial x_j}(\nabla\wedge\mathbf{u})_k \\ &= \epsilon_{ijk}\mathbf{e}_i \frac{\partial}{\partial x_j}\left(\epsilon_{kpq}\frac{\partial u_q}{\partial x_p}\right) \\ &= \epsilon_{kij}\epsilon_{kpq}\mathbf{e}_i \frac{\partial^2 u_q}{\partial x_j \partial x_p} \\ &= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\mathbf{e}_i \frac{\partial^2 u_q}{\partial x_j \partial x_p} \\ &= \mathbf{e}_i \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \mathbf{e}_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} \\ &= \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u} \end{aligned}$$

- *Example 8*: (The divergence theorem) Let the region $V$ in $\mathbb{R}^3$ be bounded by a piecewise smooth surface $\partial V$ with outward pointing unit normal $\mathbf{n} = n_j\mathbf{e}_j$. Let $\mathbf{G}(\mathbf{x}) = G_j(\mathbf{x})\mathbf{e}_j$ be a differentiable vector field on $V$. Then

$$\iint_{\partial V} \mathbf{G}\cdot\mathbf{n}\mathrm{d}S = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{G}\mathrm{d}V \quad \text{or} \quad \iint_{\partial V} G_j n_j \mathrm{d}S = \iiint_V \frac{\partial G_j}{\partial x_j}\mathrm{d}V.$$

- *Example 9*: The incompressible Navier-Stokes equations (3)-(4) may be written in the form

$$\frac{\partial u_j}{\partial x_j} = 0, \qquad \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho F_i \qquad (i = 1, 2, 3).$$

## 1.3 Kinematics

### 1.3.1 The continuum hypothesis

- Liquids and gases consist of atoms or molecules, which move around and interact with each other and with obstacles.

- At a macroscopic level, the net result of all these random interactions is that the fluid appears to be a continuous medium, or continuum.

- The continuum hypothesis is the assumption that the fluid can be characterized by properties (*e.g.* density $\rho$, velocity $\mathbf{u}$, pressure $p$, absolute temperature $T$) which depend continuously on position $\mathbf{x}$ and time $t$ (rather than having to keep track of a large number of individual atoms or molecules).

- The hypothesis holds for the vast majority of practically important flows, but can break down in extreme conditions ( *e.g.* very low density).

### 1.3.2 Eulerian and Lagrangian coordinates

- We distinguish two spatial coordinate systems, as follows.

  *Eulerian Coordinates* $\mathbf{x} = (x_1, x_2, x_3)$
  - Label points fixed in space.
  - Fluid properties at each point $\mathbf{x}$ change as different fluid particles pass through that point, *e.g.* $\mathbf{u}(\mathbf{x},\ t)$ is fluid velocity at point $\mathbf{x}$ at time $t$.
  - The Eulerian time derivative ( *i.e.* holding $\mathbf{x}$ fixed) is denoted by

$$\frac{\partial}{\partial t} \equiv \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}}.$$

  *Lagrangian Coordinates* $\mathbf{X} = (X_1,\ X_2,\ X_3)$
  - Label fluid particles and in this sense they "move with the fluid."
  - Fluid properties are described for each fluid particle as it moves through different points in space.
  - The convective or material or Lagrangian time derivative ( *i.e.* holding $\mathbf{X}$ fixed) is denoted by

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}}.$$

- We choose the label $\mathbf{X}$ to be the initial position of a fluid particle at time $t = 0$, and denote by $\mathbf{x}(\mathbf{X},\ t)$ its position at time $t \geq 0$, *i.e.*

$$\mathbf{x}(\mathbf{X},\ 0) = \mathbf{x}, \qquad \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} \mathbf{x}(\mathbf{X},\ t) = \mathbf{u}(\mathbf{x}(\mathbf{X},\ t),\ t)$$

  so that $\{\mathbf{x}(\mathbf{X},\ t) : t\ \geq 0\}$ is the pathline of the fluid particle at $\mathbf{X}$ at $t = 0$.

- The convective derivative D/Dt is related to the Eulerian time derivative $\partial/\partial t$ using the chain rule. For a differentiable scalar field $f(\mathbf{x},\ t)$,

$$
\begin{aligned}
\frac{\mathrm{D}f}{\mathrm{D}t} &= \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} f(\mathbf{x}(\mathbf{X},\ t),\ t) \\
&= \left.\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t}\right|_{\mathbf{X}} + \left.\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t}\right|_{\mathbf{X}} + \left.\frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial t}\right|_{\mathbf{X}} + \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial t} + \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\mathbf{X}} \cdot \boldsymbol{\nabla} f \\
&= \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})f \\
&= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) f.
\end{aligned}
$$

- The convective derivative may be written in the form

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}$$

and applied to vector quantities, *e.g.* the acceleration of a fluid particle is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \mathbf{e}_i\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right)$$

- Both Eulerian and Langrangian coordinates can be useful when describing fluid motion:

| | | |
|---|---|---|
| Eulerian coordinates | ⇒ | working in a fixed laboratory frame |
| | ⇒ | convenient for calculations; |
| Langrangian coordinates | ⇒ | working in the frame-of-reference of a moving fluid particle |
| | ⇒ | covenient for the application of conservation principles (e.g. mass, momentum, energy). |

### 1.3.3 The Jacobian and Euler's identity

- The continuum hypothesis implies that there is a one-to-one relation between $\mathbf{X}$ and $\mathbf{x}(\mathbf{X},\ t)$ , *i.e.* fluid can never appear from nowhere or disappear.

- We assume in addition that the map from $\mathbf{X}$ to $\mathbf{x}(\mathbf{X},\ t)$ is continuous, so that the Jacobian

$$J(\mathbf{X},\ t) = \frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} \equiv \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

is positive and bounded.

- The Jacobian $J(\mathbf{X},\ t)$ measures the change in a small volume compared with its initial volume, *i.e.*

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = J(\mathbf{X},\ t)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3,$$

with $J(\mathbf{X},\ 0) = 1$ because $\mathbf{x}(\mathbf{X},\ 0) = \mathbf{X}$ by definition.

- The Jacobian may be written in the form (using the summation convention here and hereafter)

$$J(\mathbf{X},\ t)\ = \varepsilon_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k},$$

where $\varepsilon_{ijk}$ is the Levi-Civita symbol.

- Hence, the rate of change of the Jacobian $J$ following the fluid is given by

$$\begin{aligned}
\frac{\mathrm{D}J}{\mathrm{D}t} &= \frac{\mathrm{D}}{\mathrm{D}t}\left(\varepsilon_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}\right)\\
&= \varepsilon_{ijk}\left(\frac{\partial}{\partial X_i}\left(\frac{\mathrm{D}x_1}{\mathrm{D}t}\right)\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}+\frac{\partial x_1}{\partial X_i}\frac{\partial}{\partial X_j}\left(\frac{\mathrm{D}x_2}{\mathrm{D}t}\right)\frac{\partial x_3}{\partial X_k}+\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial}{\partial X_k}\left(\frac{\mathrm{D}x_3}{\mathrm{D}t}\right)\right)\\
&= \varepsilon_{ijk}\left(\frac{\partial u_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}+\frac{\partial x_1}{\partial X_i}\frac{\partial u_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}+\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial u_3}{\partial X_k}\right)\\
&= \varepsilon_{ijk}\left(\frac{\partial u_1}{\partial x_m}\frac{\partial x_m}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}+\frac{\partial x_1}{\partial X_i}\frac{\partial u_2}{\partial x_m}\frac{\partial x_m}{\partial X_j}\frac{\partial x_3}{\partial X_k}+\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial u_3}{\partial x_m}\frac{\partial x_m}{\partial X_k}\right)\\
&= \frac{\partial u_1}{\partial x_m}\frac{\partial(x_m,x_2,x_3)}{\partial(X_1,X_2,X_3)}+\frac{\partial u_2}{\partial x_m}\frac{\partial(x_1,x_m,x_3)}{\partial(X_1,X_2,X_3)}+\frac{\partial u_3}{\partial x_m}\frac{\partial(x_1,x_2,x_m)}{\partial(X_1,X_2,X_3)}\\
&= \left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)},
\end{aligned}$$

where in the second line we used the fact $\varepsilon_{ijk}$ is constant and the product rule; in the third line we used the definition

$$\frac{\mathrm{D}x_l}{\mathrm{D}t} = u_l;$$

in the fourth line we used the chain rule to write

$$\frac{\partial u_l}{\partial X_n} = \frac{\partial u_l}{\partial x_m}\frac{\partial x_m}{\partial X_n};$$

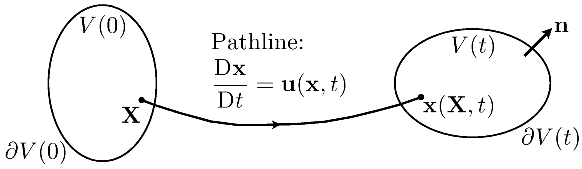
and in the sixth line we used the fact a determinant is zero if it has repeated rows.

- We have therefore derived Euler's identity

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J\boldsymbol{\nabla}\cdot\mathbf{u}. \tag{5}$$

### 1.3.4 Reynolds' transport theorem

- Consider a material volume $V(t)$ which is transported along by the fluid and bounded by the surface $\partial V(t)$ with outward unit normal $\mathbf{n}$.




- Let $f(\mathbf{x},\ t)$ be any continuously differentiable property of the fluid, *e.g.* density, kinetic energy per unit volume. The total amount of $f$ inside $V(t)$ is given by the volume integral

$$I(t)\ = \iiint_{V(t)} f(\mathbf{x},\ t)\ \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.$$

- Transforming from Eulerian coordinates $\mathbf{x}$ to Langrangian coordinates $\mathbf{X}$, the integral becomes

$$I(t)\ = \iiint_{V(0)} f(\mathbf{x}(\mathbf{X},\ t),\ t)J(\mathbf{X},\ t)\ \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3.$$

- We can now "differentiate under the integral sign" to find that the time rate of change of $I(t)$ is given by

$$
\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \iiint_{V(0)} \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} (fJ)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3 \\
&= \iiint_{V(0)} \frac{\mathrm{D}}{\mathrm{D}t}(fJ)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3 \\
&= \iiint_{V(0)} \left(\frac{\mathrm{D}f}{\mathrm{D}t}J + f\frac{\mathrm{D}J}{\mathrm{D}t}\right) \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3 \\
&= \iiint_{V(0)} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\mathbf{u}\right) J\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3 \\
&= \iiint_{V(t)} \left(\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\mathbf{u})\right) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3,
\end{aligned}
$$

where on the fourth line we used Euler's identity (ref5) and on the last line we set

$$
\frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})f + f\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\mathbf{u})
$$

while transforming back to Eulerian coordinates.

- Thus, we have proven Reynolds' transport theorem: If $V(t)$ is a material volume convected with velocity $\mathbf{u}(\mathbf{x},\ t)$ and $f(\mathbf{x},\ t)$ is a continuously differentiable function, then

$$
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f\mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\mathbf{u})\mathrm{d}V. \tag{6}
$$

Note:

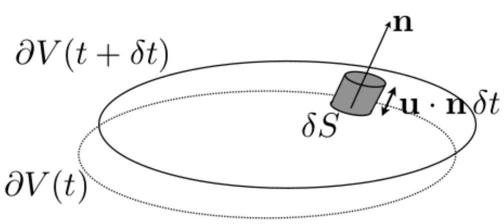
$$
\frac{d}{dt}\int_{V(t)} f(\mathbf{x},t)\,dV = \int_{V(t)} \underbrace{\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})}_{=\frac{Df}{Dt}+f\nabla\cdot\mathbf{u}}\,dV = \int_{V(t)} \frac{\partial f}{\partial t}\,dV + \int_{\partial V(t)} \underbrace{f\mathbf{u}\cdot\mathbf{n}}_{\text{flux of f through the boundary}}\,dS
$$

### 1.3.5 Visualization of Reynolds' transport theorem

- Let $\delta t$ be a small time increment, then

$$
\begin{aligned}
\frac{I(t+\delta t) - I(t)}{\delta t} &= \frac{1}{\delta t}\left(\iiint_{V(t+\delta t)} f(\mathbf{x},\ t\ +\delta t)\ \mathrm{d}V - \iiint_{V(t)} f(\mathbf{x},\ t)\ \mathrm{d}V\right) \\
&= \underbrace{\iiint_{V(t)} \frac{f(\mathbf{x},t+\delta t) - f(\mathbf{x},t)}{\delta t}\mathrm{d}V}_{\text{Change inside } V(t)} + \underbrace{\frac{1}{\delta t}\iiint_{V(t+\delta t)\setminus V(t)} f(\mathbf{x},\ t)dV}_{\text{Change due to moving boundary}}
\end{aligned}
$$

- The volume $V(t\ +\delta t)\setminus V(t)$ is a thin shell around $\partial V(t)$ . The amount of $f$ swept through a surface element $\delta S$ of $\partial V(t)$ in the time increment $\delta t$ is $f(\mathbf{u}\cdot\mathbf{n}\delta t)\delta S$:




$$
f \times \text{parcel volume} = f\,(\mathbf{u}\cdot\mathbf{n}\,\delta t)\,\delta S
$$

- Hence, as $\delta t \to 0$,

$$\frac{I(t+\delta t)-I(t)}{\delta t} \to \iiint_{V(t)} \frac{\partial f}{\partial t}\mathrm{d}V + \iint_{\partial V(t)} f\mathbf{u}\cdot\mathbf{n}\mathrm{d}S.$$

- Finally, apply the divergence theorem

$$\iint_{\partial V(t)} f\mathbf{u}\cdot\mathbf{n}\mathrm{d}S = \iiint_{V(t)} \boldsymbol{\nabla}\cdot(f\mathbf{u})\mathrm{d}V$$

to recover Reynolds' transport theorem (6).

#### 1.3.6 Conservation of mass

- Since a material volume $V(t)$ always consists of the same fluid particles, its mass must be preserved, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho\mathrm{d}V = 0.$$

- Apply Reynolds' transport theorem (6) with $f=\rho$ to obtain

$$\iiint_{V(t)} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\mathrm{d}V = 0.$$

- Since the volume $V(t)$ is arbitrary, the integrand must be zero (assuming it is continuous), *i.e.*

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0. \tag{7}$$

Or: $\quad \dfrac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad (7')$

- This equation is called the continuity equation and represents pointwise conservation of mass.
- Note that an application of Reynolds' transport theorem (6) with $f=\rho F$ implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho F\mathrm{d}V &= \iiint_{V(t)} \frac{\partial}{\partial t}(\rho F) + \boldsymbol{\nabla}\cdot(\rho F\mathbf{u})\mathrm{d}V \\ &= \iiint_{V(t)} F\underbrace{\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right)}_{=0} + \rho\underbrace{\left(\frac{\partial F}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})F\right)}_{=\frac{\mathrm{D}F}{\mathrm{D}t}}\mathrm{d}V,\end{aligned}$$

which yields the following useful corollary

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho F\mathrm{d}V = \iiint_{V(t)} \rho\frac{\mathrm{D}F}{\mathrm{D}t}\mathrm{d}V \tag{8}$$

for continuously differentiable $\rho$ and $F$.

Note that Eqn (8) may be derived directly as follows:

$$\frac{d}{dt}\int_{V(t)} F(\mathbf{x},t)\,\rho\,dV = \frac{d}{dt}\int_{V(0)} F(\mathbf{x}(\mathbf{X},t),t) \underbrace{\rho_0 dV_0}_{\substack{\rho dV=\rho_0 dV_0 \\ \text{mass conservation} \\ \text{initial configuration} \\ \text{subscript } 0}} = \int_{V(0)} \frac{DF}{Dt}\,\rho_0\,dV_0 = \int_{V(t)} \frac{DF}{Dt}\,\rho\,dV$$

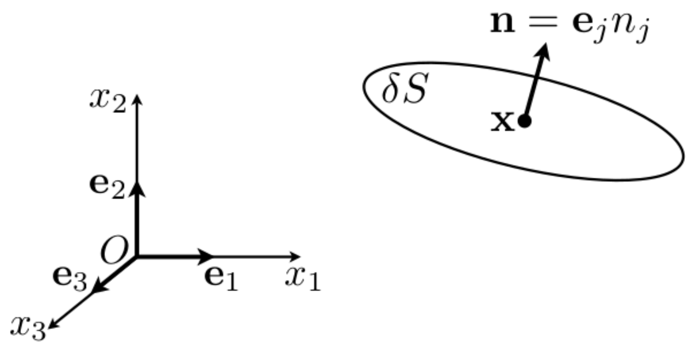
- For incompressible fluid, $\mathrm{D}\rho/\mathrm{D}t = 0$, and hence by (7) or (7') we obtain the incompressibility condition

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{9}$$

## 1.4 Dynamics

### 1.4.1 The stress vector

- Consider a surface element $\delta S$ with unit normal $\mathbf{n}$ drawn through $\mathbf{x}$ in the fluid:



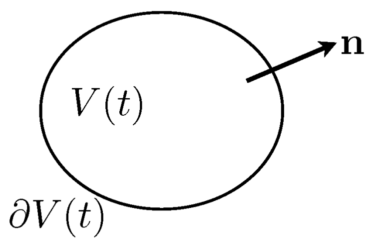

- The stress vector $\mathbf{t}(\mathbf{x}, t, \mathbf{n})$ is the force per unit area (*i.e.* stress) exerted on the surface element by the fluid toward which $\mathbf{n}$ points.

- *Example*: Fluid flows inside a rectangular box $R = \{\mathbf{x} : 0 < x_j < j \text{ for } j = 1, 2, 3\}$ whose boundary $\partial R$ is rigid and solid. The force per unit area exerted by the fluid at a point $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ on the bottom face $x_3 = 0$ is $\mathbf{t}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2,\ t,\ \mathbf{n} = \mathbf{e}_3)$, so the total force exerted by the fluid on the bottom face is given by

$$\int_0^2 \int_0^1 \mathbf{t}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2,\ t,\ \mathbf{e}_3) dx_1 dx_2.$$

### 1.4.2 Conservation of momentum

- Consider a material volume $V(t)$ with boundary $\partial V(t)$ whose outward unit normal is $\mathbf{n}$:




- Its linear momentum is $\displaystyle\iiint_{V(t)} \rho \mathbf{u} \mathrm{d}V$.

- Forces acting on $V(t)$ :

  (i) *Internal forces* represented by the stress $\mathbf{t}(\mathbf{x},\ t,\ \mathbf{n})$ exerted by the fluid outside $V(t)$ on the fluid inside $V(t)$ via the boundary $\partial V(t)$ .

  (ii) *External forces* (*e.g.* gravity, EM) represented by a body force $\mathbf{F}(\mathbf{x},\ t)$ acting per unit mass.

- Newton's second law for the material volume $V(t)$ states that the time rate of change of its linear momentum is equal to the net force applied. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \mathbf{u} \mathrm{d}V = \iint_{\partial V(t)} \mathbf{t} \mathrm{d}S + \iiint_{V(t)} \rho \mathbf{F} \mathrm{d}V. \tag{10}$$

**Example: Derivation of Euler's momentum equation**

- For an inviscid fluid the stress vector
$$\mathbf{t} = -p\mathbf{n},$$
where $p$ is the pressure.

- Note the implications:
  (i) stress is purely in the normal direction (*i.e.* no friction);
  (ii) the magnitude of the stress (i.e. $p$) is independent of the orientation of the surface element (*i.e.* of $\mathbf{n}$).

- For viscous fluid neither of these is correct: we must allow for stress which is not necessarily in the normal direction, and whose magnitude depends on $\mathbf{n}$.

- Note that the corollary to Reynolds' transport theorem (8) implies that
$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho\mathbf{u}\mathrm{d}V = \iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\mathrm{d}V,$$
while the divergence theorem implies that
$$\iint_{\partial V(t)} \mathbf{t}\mathrm{d}S = \iint_{\partial V(t)} -p\mathbf{n}\mathrm{d}S = -\mathbf{e}_i\iint_{\partial V(t)} p\delta_{ij}n_j\mathrm{d}S = -\mathbf{e}_i\iiint_{V(t)} \frac{\partial}{\partial x_j}(p\delta_{ij})\mathrm{d}V = \iiint_{V(t)} -\boldsymbol{\nabla}p\mathrm{d}V.$$

- Hence, by (10),
$$\iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \boldsymbol{\nabla}p - \rho\mathbf{F}\mathrm{d}V = 0;$$
since $V(t)$ is arbitrary, the integrand must be zero (assuming it is continuous), and we recover Euler's equation
$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{F}.$$

- In order to generalize this methodology to any continuous medium (and, in particular, a viscous fluid), it is necessary to convert the surface integral
$$\iint_{\partial V(t)} \mathbf{t}\mathrm{d}S$$
into a volume integral. This is accomplished via Cauchy's stress theorem, which recasts the stress vector in a form amenable to the divergence theorem.

### 1.4.3 The stress tensor

- The stress tensor $\sigma_{ij}(\mathbf{x},\ t)$ is the component of the stress in the $x_i$-direction exerted on a surface element with normal in the $x_j$ direction by the fluid toward which $\mathbf{e}_j$ points.

- Note that the subscript $i$ corrsponds to the direction of the stress, while the subscript $j$ corresponds to the direction of the normal. Moreover, by definition,
$$\sigma_{ij}(\mathbf{x},\ t)\ = \mathbf{e}_i \cdot \mathbf{t}(\mathbf{x},\ t,\ \mathbf{e}_j)\ ,$$
so
$$\mathbf{t}(\mathbf{x},\ t,\ \mathbf{e}_j) = \mathbf{e}_i\sigma_{ij}(\mathbf{x},\ t)\ .$$

- *Example 1*: Fluid flows in the upper half-space $x_3 > 0$ above a rigid solid plate at $x_3 = 0$. The force per unit area exerted by the fluid on the plate at a point $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ is given by
$$\mathbf{t}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2,\ t,\ \mathbf{n} = \mathbf{e}_3) = \mathbf{e}_i\sigma_{i3}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2,\ t)\ ;$$
$\sigma_{13}$ and $\sigma_{23}$ are *shear stresses*, while $\sigma_{33}$ is a normal stress.

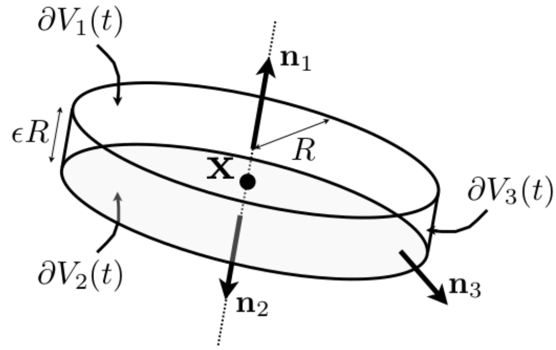
- *Example 2*: Since $\mathbf{t} = -p\mathbf{n}$ for an inviscid fluid,

$$\sigma_{ij}(\mathbf{x},\ t)\ = \mathbf{e}_i \cdot \mathbf{t}(\mathbf{x},\ t,\ \mathbf{e}_j) = \mathbf{e}_i \cdot (-p(\mathbf{x},\ t)\mathbf{e}_j) = -p(\mathbf{x},\ t)\ \delta_{ij},$$

where $\delta_{ij}$ is Kronecker's delta.

### 1.4.4 Action and reaction

- Consider a material volume $V(t)$ having at time $t$ the configuration of a right circular cylinder, with radius $R$, height $\varepsilon R$, centre $\mathbf{x}$ and outward unit normals as shown.




- Newton's second law for the material volume (10) may be written in the form

$$\iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho\mathbf{F}\mathrm{d}V = \iint_{\partial V(t)} \mathbf{t}\mathrm{d}S.$$

- Assuming the integrand is continuous (so that, in particular, the acceleration and body force are finite), the integral mean value theorem implies that

$$\iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho\mathbf{F}\mathrm{d}V = \mathrm{O}(R^3) \text{ as } R \to 0.$$

- Moreover, as $\varepsilon,\ R \to 0$,

$$\begin{aligned}\iint_{\mathbf{x}\in\partial V(t)} \mathbf{t}(\mathbf{x},\ t,\ \mathbf{n})\mathrm{d}S\ &=\ \sum_{j=1}^{3}\iint_{\mathbf{x}_j\in\partial V_j(t)} \mathbf{t}(\mathbf{x}_j,\ t,\ \mathbf{n}_j)\mathrm{d}S \\ &=\ \mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}_1)\pi R^2 + \mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}_2)\pi R^2 + \mathrm{O}(\varepsilon R^2,\ R^3)\ .\end{aligned}$$

- Combining these expressions gives

$$(\mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}_1) + \mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}_2))R^2 = \mathrm{O}(\varepsilon R^2,\ R^3) \text{ as } \varepsilon,\ R \to 0.$$

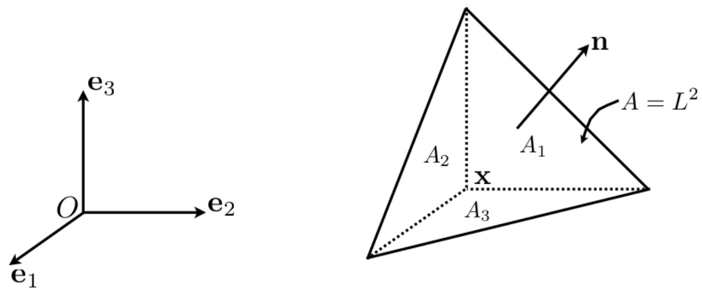
- Since this expression pertains for arbitrarily small $\varepsilon$ and $R$, we deduce (setting $\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$)

$$\mathbf{t}(\mathbf{x},\ t,\ -\mathbf{n}) = -\mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}), \tag{11}$$

which is Newton's third law (action-reaction) for a continuous medium.

### 1.4.5 Cauchy's stress theorem

- Consider a material volume $V(t)$ having at time $t$ the configuration of a small tetrahedron as shown. Let the slanting face have area $A = L^2$ and outward unit normal $\mathbf{n} = \mathbf{e}_j n_j$, with $n_j > 0$.




- Newton's second law for the material volume (10)

$$\iiint_{V(t)} \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho \mathbf{F} \mathrm{d}V = \iint_{\partial V(t)} \mathbf{t} \mathrm{d}S.$$

- Assuming the integrand is continuous, the integral mean value theorem implies that

$$\iiint_{V(t)} \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho \mathbf{F} \mathrm{d}V = \mathrm{O}(L^3) \text{ as } L \to 0.$$

- Since the face with area $A_j = n_j A = n_j L^2$ (by Q1(b)) has outward unit normal $-\mathbf{e}_j$ and the slanted face with area $A = L^2$ has outward unit normal $\mathbf{n}$,

$$\iint_{\partial V(t)} \mathbf{t} \mathrm{d}S = (\mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}) + \mathbf{t}(\mathbf{x},\ t,\ -\mathbf{e}_j) n_j) L^2 + \mathrm{O}(L^3) \text{ as } L \to 0.$$

- Combining these expressions and using Newton's third law (11) gives

$$(\mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}) - \mathbf{t}(\mathbf{x},\ t,\ \mathbf{e}_j) n_j) L^2 = \mathrm{O}(L^3) \text{ as } L \to 0.$$

- This expression pertains for arbitrarily small $L$, so there is a local equilibrium of the surface stresses, with

$$\mathbf{t}(\mathbf{x},\ t,\ \mathbf{n}) = \mathbf{t}(\mathbf{x},\ t,\ \mathbf{e}_j) n_j.$$

- Note that this expression holds for an arbitrarily oriented unit normal $\mathbf{n}$ by a straightforward generalization of the above argument.

- Finally, since $\mathbf{t}(\mathbf{x},\ t,\ \mathbf{e}_j) = \mathbf{e}_i \sigma_{ij}(\mathbf{x},\ t)$ by definition, we deduce Cauchy's stress theorem

$$\mathbf{t}(\mathbf{n}) = \mathbf{e}_i \sigma_{ij} n_j \tag{12}$$

  where we have suppressed the dependence of $\mathbf{t}$ and $\sigma_{ij}$ on $\mathbf{x}$ and $t$.

- Thus, knowing the nine quantities $\sigma_{ij}$ we can compute the stress in any direction.

### 1.4.6 Cauchy's momentum equation

- We return to the conservation of momentum of a material volume (10).
- Recall that the corollary to Reynolds' transport theorem (8) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho\mathbf{u}\mathrm{d}V = \iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\mathrm{d}V.$$

- Using Cauchy's Stress theorem (12), the net surface force is

$$\iint_{\partial V(t)} \mathbf{t}(\mathbf{n})\mathrm{d}S = \mathbf{e}_i\iint_{\partial V(t)} \sigma_{ij}n_j\mathrm{d}S = \mathbf{e}_i\iiint_{V(t)} \frac{\partial\sigma_{ij}}{\partial x_j}\mathrm{d}V$$

  after an application of the divergence theorem.
- Combining these expressions we find that (10) may be written in the form

$$\iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathbf{e}_i\frac{\partial\sigma_{ij}}{\partial x_j} - \rho\mathbf{F}\mathrm{d}V = 0.$$

- Since $V(t)$ is arbitrary, the integrand must be zero (if it is continuous), and we deduce Cauchy's momentum equation

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{e}_i\frac{\partial\sigma_{ij}}{\partial x_j} + \rho\mathbf{F}, \tag{13}$$

  which holds for any continuum, not just a fluid.

### 1.4.7 Symmetry of the stress tensor

- For a material volume $V(t)$ , conservation of angular momentum about the origin $O$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathbf{x}\wedge\rho\mathbf{u}\mathrm{d}V = \iint_{\partial V(t)} \mathbf{x}\wedge\mathbf{t}\mathrm{d}S + \iiint_{V(t)} \mathbf{x}\wedge\rho\mathbf{F}\mathrm{d}V. \tag{14}$$

- Applying Reynolds' transport theorem and the divergence theorem to this expression gives

$$\iiint_{V(t)} \mathbf{x}\wedge\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathbf{e}_i\frac{\partial\sigma_{ij}}{\partial x_j} - \rho\mathbf{F}\right)\mathrm{d}V = \iiint_{V(t)} \mathbf{e}_j\wedge\mathbf{e}_i\sigma_{ij}\mathrm{d}V.$$

- We can then deduce from Cauchy's momentum equation (13) that

$$\iiint_{V(t)} \mathbf{e}_j\wedge\mathbf{e}_i\sigma_{ij}\mathrm{d}V = 0.$$

- Since $V(t)$ is arbitrary, the integrand must be zero (if it is continuous), *i.e.*

$$0 = \mathbf{e}_j\wedge\mathbf{e}_i\sigma_{ij} = \mathbf{e}_1(\sigma_{32}-\sigma_{23}) + \mathbf{e}_2(\sigma_{13}-\sigma_{31}) + \mathbf{e}_3(\sigma_{21}-\sigma_{12}) \ ,$$

  which implies that the stress tensor is symmetric, *i.e.*

$$\sigma_{ij} = \sigma_{ji}.$$

- The stress tensor may also be shown to be symmetric by taking $V(t)$ to be instantaneously a vanishingly small cube in (14) and estimating the various terms as in the derivations of Newton's third law (11) and Cauchy's stress theorem (12).
- Note that the symmetry of the stress tensor $\sigma_{ij}$ implies that it consists of six independent quantities only, namely $\sigma_{11},\ \sigma_{22},\ \sigma_{33},\ \sigma_{12}=\sigma_{21},\ \sigma_{13}=\sigma_{31}$ and $\sigma_{23}=\sigma_{32}$.

**Alternative notes for sections 1.4.1–1.4.7**

## 1.4.1 The stress vector

Within a fluid, take a plane with unit normal vector $\mathbf{n}$. Let $\mathbf{t}(\mathbf{x}, t, \mathbf{n})$ denote the stress (force per unit area) that the *fluid toward which* $\mathbf{n}$ points, exerts on the plane at location $\mathbf{x}$ and time $t$. The vector $\mathbf{t}$ is called the stress (or traction) vector.

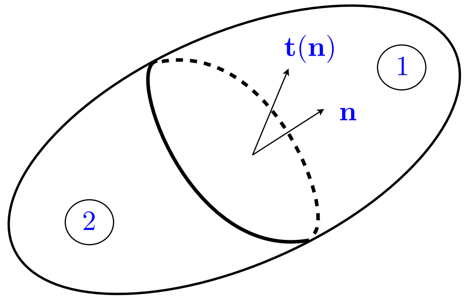



Figure 1: The stress or traction vector $\mathbf{t}(\mathbf{n})$ is the force per unit area that the fluid (material) in ① exerts on the fluid (material) in ②.

## 1.4.2 - 1.4.7 Balance Laws and the Stress vector

We distinguish between two types of forces: body forces and surface forces (tractions).

Body forces: These act at each point within the fluid (continuum) and are due to some external cause e.g. gravity. Let $\mathbf{F}$ be the body force per unit mass. Then the total body force on a fluid (continuum) occupying a volume $V(t)$ at time $t$ is

$$\int_{V(t)} \mathbf{F}\,\rho\,dV.$$

The total moment of the body forces about the origin is

$$\int_{V(t)} \mathbf{x}\wedge\mathbf{F}\,\rho\,dV,$$

where $\mathbf{x}$ is the position vector of a material point in time $t$ relative to the origin O.

Surface forces: These act across a surface and may respresent a traction (load) applied on an outer boundary, or may act across an internal surface e.g. fluid pressure. Let $\mathbf{t}$ denote the force vector per unit area (stress vector) acting on the surface $\partial V(t)$ of the fluid (continuum) at time $t$. The total surface force on $\partial V(t)$ is

$$\int_{\partial V(t)} \mathbf{t}\,dS$$

and the total moment of the surface forces about the origin is

$$\int_{\partial V(t)} \mathbf{x}\wedge\mathbf{t}\,dS.$$

For a fluid (continuum) in motion, the forces and their moments must equal the rate of change of linear and angular momentum respectively.
The linear momentum balance law:

$$\frac{d}{dt}\int_{V(t)} \mathbf{u}\rho dV = \int_{V(t)} \mathbf{F}\,\rho\,dV + \int_{\partial V(t)} \mathbf{t}\,dS \tag{10}$$

or (using Reynold's Transport Theorem or arguing directly by transforming to the original configuration)

$$\int_{V(t)} \frac{D\mathbf{u}}{Dt}\,\rho\,dV = \int_{V(t)} \mathbf{F}\,\rho\,dV + \int_{\partial V(t)} \mathbf{t}\,dS. \tag{10'}$$

The angular momentum balance law:

$$\frac{d}{dt}\int_{V(t)} \mathbf{x}\wedge\mathbf{u}\,\rho\,dV = \int_{V(t)} \mathbf{x}\wedge\mathbf{F}\,\rho\,dV + \int_{\partial V(t)} \mathbf{x}\wedge\mathbf{t}\,dS \tag{14}$$

or equivalently

$$\int_{V(t)} \mathbf{x} \wedge \frac{D\mathbf{u}}{Dt} \rho \, dV = \int_{V(t)} \mathbf{x} \wedge \mathbf{F} \rho \, dV + \int_{\partial V(t)} \mathbf{x} \wedge \mathbf{t} \, dS, \tag{14'}$$

using that $\frac{D}{Dt}(\mathbf{x} \wedge \mathbf{u}) = \underbrace{\frac{D\mathbf{x}}{Dt} \wedge \mathbf{u}}_{= 0 \text{ since } \frac{D\mathbf{x}}{Dt} = \mathbf{u}} + \mathbf{x} \wedge \frac{D\mathbf{u}}{Dt}$

The Cauchy Stress Principle:

Consider an infinitesimal tetrahedron. The outward unit normal vectors wrt the faces of the tetrahedron are

$$-\mathbf{e}_1, \quad -\mathbf{e}_2, \quad -\mathbf{e}_3, \quad \mathbf{n},$$

with corresponding areas being

$$\delta S_1, \quad \delta S_2, \quad \delta S_3, \quad \delta S.$$

By the divergence theorem

$$0 = \int_{\partial V} \mathbf{n} \, dS = -\mathbf{e}_1 \delta S_1 - \mathbf{e}_2 \delta S_2 - \mathbf{e}_3 \delta S_3 + \mathbf{n} \delta S.$$

[ Divergence theorem: For a constant vector $\mathbf{a}$, $\nabla \cdot \mathbf{a} = 0$ and thus

$$0 = \int_V \nabla \cdot \mathbf{a} \, dV = \int_{\partial V} \mathbf{a} \cdot \mathbf{n} \, dS = \mathbf{a} \cdot \int_{\partial V} \mathbf{n} \, dS.$$

$$\mathbf{a} \text{ is arbitrary} \implies \int_{\partial V} \mathbf{n} \, dS = 0 \qquad ]$$

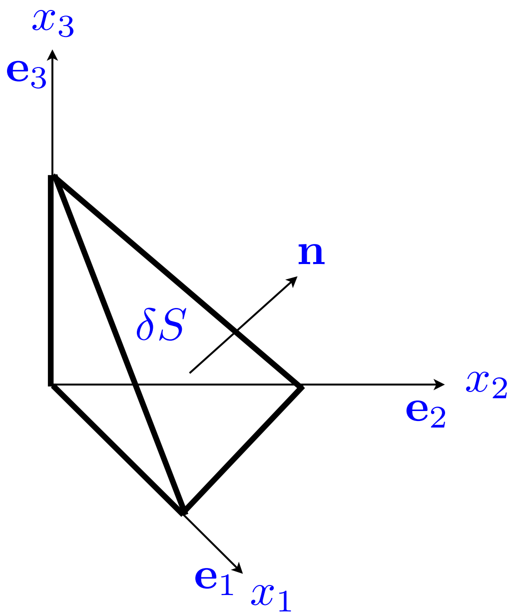



Figure 2: Infinitesimal tetrahedron.

Taking the dot product with $\mathbf{e}_i$ : $\quad \delta S_i = (\mathbf{n} \cdot \mathbf{e}_i) \delta S = n_i \delta S$

The tractions on the faces of the tetrahedron are

$$\mathbf{t}(-\mathbf{e}_1), \quad \mathbf{t}(-\mathbf{e}_2), \quad \mathbf{t}(-\mathbf{e}_3), \quad \mathbf{t}(\mathbf{n}), \quad \text{respectively,}$$

the dependence on $\mathbf{x}$ and $t$ being omitted. The linear momentum balance law can be written as

$$\int_{\partial V} \mathbf{t} \, dS = \int_V \left( \frac{D\mathbf{u}}{Dt} - \mathbf{F} \right) \rho \, dV.$$

Since the tetrahedron is infinitesimal, quantities may be regarded as constants to a leading order approximation. Thus

$$\mathbf{t}(-\mathbf{e}_1)\delta S_1 + \mathbf{t}(-\mathbf{e}_2)\delta S_2 + \mathbf{t}(-\mathbf{e}_3)\delta S_3 + \mathbf{t}(\mathbf{n})\delta S = \left( \frac{D\mathbf{u}}{Dt} - \mathbf{F} \right) \rho \, \delta V$$

or

$$\mathbf{t}(\mathbf{n}) = -\mathbf{t}(-\mathbf{e}_1) n_1 - \mathbf{t}(-\mathbf{e}_2) n_2 - \mathbf{t}(-\mathbf{e}_3) n_3 + \left( \frac{D\mathbf{u}}{Dt} - \mathbf{F} \right) \rho \, \frac{\delta V}{\delta S}.$$

Since $\delta V / \delta S \to 0$ as $\delta V \to 0$, then in the limit we have

$$\mathbf{t}(\mathbf{n}) = -\mathbf{t}(-\mathbf{e}_i) \, n_i$$

But $\mathbf{t}(\mathbf{e}_i) = -\mathbf{t}(-\mathbf{e}_i)$, which follows from Newton's Third Law or formally by taking $\mathbf{n} = \mathbf{e}_j$ so that $\mathbf{t}(\mathbf{e}_j) = -\mathbf{t}(-\mathbf{e}_i)\delta_{ij} = -\mathbf{t}(-\mathbf{e}_j)$. We thus obtain

The Cauchy Stress Principle:

$$\mathbf{t}(\mathbf{n}) = \mathbf{t}(\mathbf{e}_j)n_j$$

or in components (scalar form)

$$t_i(\mathbf{n}) = t_i(\mathbf{e}_j)n_j.$$

Definition: The Cauchy Stress tensor is defined by

$$\sigma_{ij} = t_i(\mathbf{e}_j)$$

so that

$$\mathbf{t} = \mathbf{e}_i\sigma_{ij}n_j \qquad \text{or} \qquad t_i = \sigma_{ij}n_j. \tag{12}$$

Notes:

- $\mathbf{t}$ and $\mathbf{n}$ are vectors, the Quotient Rule for tensors $\implies$ $(\sigma_{ij})$ is CT2 (Cartesian Tensor of rank 2).
- $\sigma_{ij}$ is the $i-$component of the traction on a surface element with normal in the $\mathbf{e}_j$ direction.

$$(\sigma_{ij}) = \begin{pmatrix} t_1(\mathbf{e}_1) & | & t_1(\mathbf{e}_2) & | & t_1(\mathbf{e}_3) \\ t_2(\mathbf{e}_1) & | & t_2(\mathbf{e}_2) & | & t_2(\mathbf{e}_3) \\ t_3(\mathbf{e}_1) & | & t_3(\mathbf{e}_2) & | & t_3(\mathbf{e}_3) \end{pmatrix}$$

- The diagonal components $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are called the normal stresses and $\sigma_{ij}, i \neq j$ are the shear stresses.

Example: On a plane element with normal $\mathbf{e}_1$ the stress components are $t_i(\mathbf{e}_1) = \sigma_{i1}$.

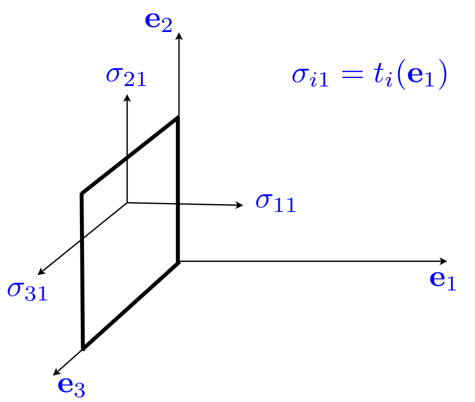



Figure 3: Plane element with normal $\mathbf{e}_1$

By the divergence theorem we have that

$$\int_{\partial V(t)} \mathbf{t}\, dS = \mathbf{e}_i \int_{\partial V(t)} \sigma_{ij}n_j\, dS = \mathbf{e}_i \int_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j}\, dV$$

and thus the linear momentum balance law (10') becomes

$$\int_{V(t)} \left(\rho\mathbf{F} + \mathbf{e}_i\frac{\partial\sigma_{ij}}{\partial x_j} - \rho\frac{D\mathbf{u}}{Dt}\right)\, dV = 0.$$

Since V is arbitray and the integrand is continuous we may deduce Cauchy's momentum equation

$$\rho\frac{D\mathbf{u}}{Dt} = \mathbf{e}_i\frac{\partial\sigma_{ij}}{\partial x_j} + \rho\mathbf{F}. \tag{13}$$

Similarly for the angular momentum balance law (14'), we have using the divergence theorem that

$$\int_{\partial V(t)} \mathbf{x}\wedge\mathbf{t}\, dS = (\mathbf{e}_k\wedge\mathbf{e}_i)\int_{\partial V(t)} x_k\sigma_{ij}n_j\, dS = (\mathbf{e}_k\wedge\mathbf{e}_i)\int_{V(t)} \frac{\partial(x_k\sigma_{ij})}{\partial x_j}\, dV = (\mathbf{e}_k\wedge\mathbf{e}_i)\int_{V(t)} \underbrace{\frac{\partial x_k}{\partial x_j}}_{=\delta_{kj}}\sigma_{ij} + x_k\frac{\partial\sigma_{ij}}{\partial x_j}\, dV$$

$$= \int_{V(t)} (\mathbf{e}_j\wedge\mathbf{e}_i)\sigma_{ij} + \mathbf{x}\wedge\mathbf{e}_i\frac{\partial\sigma_{ij}}{\partial x_j}\, dV$$

and hence (14') becomes

$$\int_{V(t)} \mathbf{x} \wedge \underbrace{\left(\rho \frac{D\mathbf{u}}{Dt} - \mathbf{e}_i \frac{\partial \sigma_{ij}}{\partial x_j} - \rho \mathbf{F}\right)}_{=0 \text{ using (13)}} dV = \int_{V(t)} \mathbf{e}_j \wedge \mathbf{e}_i \sigma_{ij} dV.$$

Again as $V$ is arbitrary and the integrand continuous we deduce that

$$\mathbf{e}_j \wedge \mathbf{e}_i \sigma_{ij} = 0 \qquad \text{or} \qquad \epsilon_{kji} \mathbf{e}_k \sigma_{ij} = 0.$$

Hence

$$\mathbf{e}_1(\sigma_{23} - \sigma_{32}) + \mathbf{e}_2(\sigma_{13} - \sigma_{31}) + \mathbf{e}_3(\sigma_{12} - \sigma_{21}) = 0$$

from which it follows that $\sigma_{ij} = \sigma_{ji}$ and the Cauchy stress tensor is symmetric. Alternatively,

$$\epsilon_{kji}\sigma_{ij} = 0 \implies \epsilon_{kpq}\epsilon_{kji}\sigma_{ij} = 0 \implies (\delta_{pi}\delta_{qj} - \delta_{pj}\delta_{qi})\sigma_{ij} = 0 \implies \sigma_{pq} - \sigma_{qp} = 0.$$

### 1.4.8 Change of coordinate system

- Suppose we rotate the coordinate system from $Ox_1x_2x_3$ with orthonormal basis vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ to $Ox_1'x_2'x_3'$ with orthonormal basis vectors $\{\mathbf{e}_1',\ \mathbf{e}_2',\ \mathbf{e}_3'\}$.

- A position vector $\mathbf{r}$ may be written $\mathbf{r} = x_j\mathbf{e}_j = x_i'\mathbf{e}_i'$, so the rotation of the coordinate system transforms the coordinates of the vector $\mathbf{r}$ according to

$$x_i' = \mathbf{r}\cdot\mathbf{e}_i' = (x_j\mathbf{e}_j)\cdot\mathbf{e}_i' = l_{ij}x_j, \tag{15}$$

where $l_{ij} = \mathbf{e}_i'\cdot\mathbf{e}_j$.

- Equally, we can write $\mathbf{r} = x_i\mathbf{e}_i = x_j'\mathbf{e}_j'$ and deduce that the inverse transformation is given by

$$x_i = \mathbf{r}\cdot\mathbf{e}_i = (x_j'\mathbf{e}_j')\cdot\mathbf{e}_i = l_{ji}x_j'. \tag{16}$$

- Combining (15) and (16) gives

$$x_i \ = l_{ji}x_j' = l_{ji}l_{jk}x_k \equiv \delta_{ik}x_k,$$

where $\delta_{ik}$ is Kronecker's delta.

- Since the last expression holds for arbitrary $x_k$, we deduce that

$$l_{ji}l_{jk} = \delta_{ik},$$

*i.e.* the matrix $L = \{l_{ij}\}_{3\times3}$ is orthogonal, with

$$LL^T = I = L^T L,$$

where $I = \{\delta_{ij}\}_{3\times3}$ is the 3-by-3 identity matrix.

- By definition the stress tensor in the primed frame is

$$\sigma_{rs}' = \mathbf{e}_r'\cdot\mathbf{t}(\mathbf{e}_s')\ .$$

- Writing Cauchy's stress theorem (12) in the original frame in the form

$$\mathbf{t}(\mathbf{n}) = \mathbf{e}_i\sigma_{ij}n_j = \mathbf{e}_i\sigma_{ij}(\mathbf{n}\cdot\mathbf{e}_j)\ ,$$

we deduce that under the rotation of the coordinate system due to the orthogonal matrix $L = \{l_{ij}\}_{3\times3}$, the stress tensor transforms according to

$$\sigma_{rs}' = \mathbf{e}_r'\cdot\big(\mathbf{e}_i\sigma_{ij}(\mathbf{e}_s'\cdot\mathbf{e}_j)\big) = l_{ri}l_{sj}\sigma_{ij}, \tag{17}$$

or equivalently

$$S' = LSL^T,$$

where

$$S' = \{\sigma_{rs}'\}_{3\times3},\ S = \{\sigma_{ij}\}_{3\times3},$$

- That $\sigma_{ij}$ transforms according to (17) means that it is a *second-rank tensor*, which are a generalization of vector fields or first-rank tensors, *cf.* (15) and (17).

- It is for this reason that $\sigma_{ij}$ is called the stress tensor and the upshot of the above analysis is that there is an invariantly defined stress in the fluid.

- Note that $\sigma_{ij}$ is a second-rank tensor if and only if Cauchy's stress theorem is independent of the choice of of coordinate system, *i.e.* (17) holds iff

$$\mathbf{t}'(\mathbf{n}') = \mathbf{e}_r'\sigma_{rs}'n_s',$$

where $\mathbf{t}' = t_r'\mathbf{e}_r',\ \mathbf{n}' = n_s'\mathbf{e}_s'$, with $t_r' = l_{ri}t_i,\ n_s' = l_{sj}n_j$.

## 1.5 Newtonian constitutive law

### 1.5.1 Recap

- We have derived for a continuous medium expressions (7) and (13) representing conservation of mass and momentum, *viz.*

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \qquad \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{e}_i \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \mathbf{F}.$$

- The number of scalar equations ($1{+}3 = 4$) is less than the number of unknowns ($\rho, \mathbf{u},\ \sigma_{ij} = \sigma_{ji},\ i.e.\ 1{+}3{+}6 = 10$) , so we need more information to close the system.

### 1.5.2 Constitutive relations

- To make progress we must decide how the stress tensor $\sigma_{ij}$ depends on the pressure $p$ and velocity $\mathbf{u}$.
- This is called a constitutive relation and cannot be deduced, relying instead on some assumptions about the physical properties of the material under consideration.
- For example, we would expect the constitutive relation for a solid to be quite different from that for a fluid.
- Examples of simple constitutive relations:
  (i) Hooke's law for the extension of a spring;
  (ii) Fourier's law for the flux of heat energy down the temperature gradient;
  (iii) the inviscid stress tensor $\sigma_{ij} = -p\delta_{ij}$.
- Note that
  (i) a "thought-experiment" suggests these laws are reasonable;
  (ii) they could be confirmed experimentally;
  (iii) they will almost certainly fail under "extreme" conditions.

**Example: Heat conduction in a stationary isotropic continuous medium**

- Let $T(\mathbf{x},\ t)$ be the absolute temperature in a stationary isotropic continuous medium (*e.g.* a fluid or a rigid solid at rest), with constant density $\rho$ and specific heat $c_v$.
- Let $\mathbf{q}(\mathbf{x},\ t)$ be the heat flux vector, so that $\mathbf{q} \cdot \mathbf{n}$ is the rate of transport of heat energy per unit area across a surface element in the direction of its unit normal $\mathbf{n}$.
- For a fixed region $V$ in the medium with boundary $\partial V$ whose outward unit normal is $\mathbf{n}$, conservation of heat energy is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho c_v T \mathrm{d}V = \iint_{\partial V} \mathbf{q} \cdot (-\mathbf{n})\ \mathrm{d}S,$$

where the term on the left-hand side (LHS) is the rate of increase of internal heat energy and the term on the right-hand side (RHS) is the rate of heat conduction into $V$ across $\partial V$.

- Differentiating under the integral sign on the LHS and applying the divergence theorem on the RHS gives

$$\iiint_V \left( \rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{q} \right) \mathrm{d}V = 0.$$

- Since $V$ is arbitrary, the integrand must be zero (if it is continuous), *i.e.*

$$\rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{q} = 0.$$

- A closed model for heat conduction is obtained by prescribing a constitutive law relating the heat flux vector $\mathbf{q}$ to the temperature $T$.

- Fourier's Law states that heat energy is transported down the temperature gradient, with

$$\mathbf{q} = -k\boldsymbol{\nabla}T,$$

  where $k$ is the constant thermal conductivity.

- Hence, $T$ satisfies the heat or diffusion equation

$$\frac{\partial T}{\partial t} = \kappa\boldsymbol{\nabla}^2 T,$$

  where the thermal diffusivity $\kappa = k/\rho c_v$.

- The SI units of the dependent variables and dimensional parameters are summarized in the following table; note that kelvin $K$ is the SI unit of temperature, joule J is the SI derived unit of energy (1J = 1Nm) and the newton N is the SI derived unit of force (1N = 1Kgms$^{-1}$) .

| Quantity | Symbol | SI units |
|---|---|---|
| Temperature | $T$ | K |
| Heat flux vector | $\mathbf{q}$ | Jm$^{-2}$s$^{-1}$ |
| Density | $\rho$ | Kgm$^{-3}$ |
| Specific heat | $c_v$ | JKg$^{-1}$k$^{-1}$ |
| Thermal conductivity | $k$ | Jm$^{-1}$s$^{-1}$K$^{-1}$ |
| Thermal diffusivity | $\kappa$ | m$^2$s$^{-1}$ |

### 1.5.3 The Couette flow rheometer

- A layer of viscous fluid of height $h$ is sheared between two parallel plates by moving the top plate horizontally with speed $U$.

- The force required to maintain the motion of the top plate is proportional to $U$ and inversely proportional $h$.

- Thus, the shear stress exerted by the top plate on the fluid must satisfy

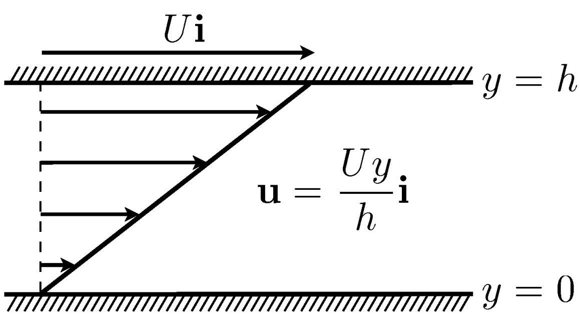
$$\sigma_{12}\big|_{y=h} \propto \frac{U}{h}.$$

- The liquid flows parallel to the plates, with a velocity profile that is linear in $y$.

- These observations suggest a constitutive law of the form

$$\sigma_{12} = \mu\frac{\partial u}{\partial y}, \qquad (18)$$

  where $\mu$ is a constant of proportionality that depends on the liquid only (in fact $\mu$ is the dynamic viscosity).



### 1.5.4 The Newtonian constitutive law

- To generalize (18), we begin by writing
$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \tag{19}$$
where $-p\delta_{ij}$ is the inviscid stress tensor and $\tau_{ij}$ is the deviatoric stress tensor, due to the presence of viscosity.

- Experiments suggest that

  (A) $\tau_{ij}$ is a linear function of the velocity gradients $\partial u_\alpha/\partial x_\beta$;

  (B) the relation between $\tau_{ij}$ and the velocity gradients is isotropic, *i.e.* invariant to rotations of the coordinate axes (so that there is no preferred direction).

- These conditions define a Newtonian fluid because they are sufficient to determine the form of $\tau_{ij}$ completely: together with symmetry of $\sigma_{ij}$, (A)–(B) imply that
$$\tau_{ij} = \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\delta_{ij} + 2\mu e_{ij}, \tag{20}$$
where $\lambda$ is the bulk viscosity, $\mu$ is the dynamic (shear) viscosity and
$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$
is the rate-of-strain tensor (which is zero for any rigid-body motion, *i.e.* if there is no deformation of fluid elements).

- The expression (20) is the constitutive law for a Newtonian fluid.

**Outline Proof**

- By property (A),
$$\tau_{ij} = A_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta},$$
where $A_{ij\alpha\beta}$ are constants - 81 of them!

- Since $\tau_{ij}$ and $\partial u_\alpha/\partial x_\beta$ are rank-2 tensors, tensor theory implies that $A_{ij\alpha\beta}$ is a rank-4 tensor.

- Property (B) means that $A_{ij\alpha\beta}$ is an isotropic tensor, *i.e.* if $\tau'_{ij} = A'_{ij\alpha\beta}\partial u'_\alpha/\partial x'_\beta$ under rotation of the coordinate system due to the orthogonal matrix $L = \{l_{ij}\}_{3\times 3}$ in §1.4.8, then $A'_{ij\alpha\beta} = A_{ij\alpha\beta}$.

- Since $A_{ij\alpha\beta}$ is an isotropic rank-4 tensor, tensor theory implies that
$$A_{ij\alpha\beta} = \lambda\delta_{ij}\delta_{\alpha\beta} + \mu(\delta_{i\alpha}\delta_{j\beta} + \delta_{i\beta}\delta_{j\alpha}) + \mu^\dagger(\delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha})\ ,$$
where $\mu$, $\mu^\dagger$ and $\lambda$ are constants-just 3 of them!

- Since $\tau_{ij}$ is symmetric, $A_{ij\alpha\beta} = A_{ji\alpha\beta}$, which implies $\mu^\dagger = 0$, and hence that
$$\tau_{ij} = A_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij}.$$

- Finally, note that $e_{\alpha\alpha} = \partial u_\alpha/\partial x_\alpha = \boldsymbol{\nabla}\cdot\mathbf{u}$.

**Summary:**
Macroscale: Sliding one plate over another

$$\sigma_{12} \propto \frac{U}{h}.$$

Assume that this applies at the microscale to layers of fluid sliding over one another

$$\sigma_{12} \propto \lim_{\delta y \to 0} \frac{u(y+\delta y) - u(y)}{\delta y} = \frac{\partial u}{\partial y}.$$

We could define the hydrodynamic pressure $p$ to be

$$p = -\frac{1}{3}\sigma_{kk} = -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}),$$

so that $\tau_{kk} = 0$. As such (20) gives

$$0 = (3\lambda + 2\mu)e_{kk}$$

and hence the bulk viscosity $\lambda = -\frac{2}{3}\mu$. The constitutive law for a Newtonian fluid is thus

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(e_{ij} - \tfrac{1}{3}e_{kk}\delta_{ij}).$$

Definition: The scalar parameter $\mu$ is called the dynamic viscosity of the fluid. It is the proportionality factor between the rate of shear and the tangential force per unit area when plane layers of fluid slide over each other.
Typical values of $\mu$ for some common fluids are given in Batchelor and Acheson. It charasterises resistance to shear and has dimensions of [stress x time] = $\mathrm{MLT^{-2}L^{-2}T} = \mathrm{ML^{-1}T^{-1}}$ and units of $\mathrm{kg/m\,s}$ = Pa s (Pascal second, 1Pa= $1\mathrm{Nm^{-2}}$). It can vary with temperature, pressure and density, but will be assumed constant.

#### 1.5.5 Incompressibility assumption

- Except where stated we will assume that the density $\rho$ is constant, so that
  - the flow is incompressible;
  - the continuity equation (7) is replaced by the incompressibility condition (9);
  - the Newtonian constitutive law (20) becomes

$$\tau_{ij} = 2\mu e_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{21}$$

    so that the bulk viscosity $\lambda$ drops out of the model.

- Note that
  - most liquids are virtually incompressible except at extremely high pressures;
  - most gases are compressible, but the effects of compressibility are negligible at speeds well below the sound speed.

- In general, the viscosity $\mu$ may depend on the state variables, *e.g.* $\rho$, $\mathbf{u}$, $p$ or $T$, but we will take it to be constant.

### 1.6 The Navier-Stokes equations

- For an incompressible Newtonian viscous fluid (19) and (21) give

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

- We calculate

$$\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) \\
&= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_j \partial x_i} \\
&= -\frac{\partial p}{\partial x_i} + \mu\boldsymbol{\nabla}^2 u_i + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \\
&= -\frac{\partial p}{\partial x_i} + \mu\boldsymbol{\nabla}^2 u_i + \mu\frac{\partial}{\partial x_i}(\boldsymbol{\nabla}\cdot\mathbf{u}) \\
&= -\frac{\partial p}{\partial x_i} + \mu\boldsymbol{\nabla}^2 u_i
\end{aligned}$$

since for an incompressible fluid,

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0. \tag{22}$$

- Hence, we have derived from Cauchy's momentum equation (13) the incompressible Navier-Stokes equation

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\mathbf{u} + \rho\mathbf{F}. \tag{23}$$

**Remarks**

(i) Note that the incompressible Navier-Stokes equations (22)-(23) consist of four scalar equations, which is the same as the number of unknowns $(u_1,\ u_2,\ u_3,\ p)$.

(ii) In Cartesian coordinates $Ox_1x_2x_3$,

$$\mathbf{u}\cdot\boldsymbol{\nabla} = u_j\frac{\partial}{\partial x_j}, \qquad \boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x_j \partial x_j},$$

so 22)-(23) are given in component form by

$$\frac{\partial u_j}{\partial x_j} = 0, \qquad \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho F_i \quad (i = 1,2,3)\ .$$

In other coordinate systems ( *e.g.* cylindrical and spherical polar coordinates) the basis vectors themselves depend on the coordinates. To calculate the components of the momentum equation in the direction of the basis vectors, use the identities

$$\begin{aligned}
(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} &= (\boldsymbol{\nabla}\wedge\mathbf{u})\wedge\mathbf{u} + \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{u}|^2\right), \\
\boldsymbol{\nabla}^2\mathbf{u} &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u})
\end{aligned}$$

to write the momentum equation in the form

$$\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\ \wedge\mathbf{u} + \boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2\right) = -\nu\boldsymbol{\nabla}\wedge\boldsymbol{\omega} + \mathbf{F},$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\wedge\mathbf{u}$ is the vorticity and $\nu = \mu/\rho$ is the kinematic viscosity; then use the usual expressions for $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}\wedge$ in the relevant coordinate system (*e.g.* see Acheson Appendices A.6 and A.7 for the Navier-Stokes equations in cylindrical and spherical polar coordinates).

(iii) The force exerted by the fluid on a solid boundary $S$ with unit normal $\mathbf{n}$ pointing into the fluid is given by

$$\iint_S \mathbf{t}(\mathbf{n})\ \mathrm{d}S.$$

For an incompressible Newtonian fluid the stress vector $\mathbf{t}(\mathbf{n})$ may be written in the form

$$\mathbf{t}(\mathbf{n}) = -p\mathbf{n} + \mu\left[2(\mathbf{n}\cdot\boldsymbol{\nabla})\mathbf{u} + \mathbf{n}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u})\right],$$

which quantifies the remarks made in the example in §1.4. Care must be taken to evaluate the term $(\mathbf{n}\cdot\boldsymbol{\nabla})\mathbf{u}$ for coordinate systems that are not Cartesian. In this course we will work almost exclusively in Cartesian coordinates.

(iv) The Navier-Stokes equations (*i.e.* (22)–(23) with $\mu > 0$) are of higher order than the Euler equations (*i.e.* (22)–(23) with $\mu = 0$) by virtue of the "diffusive" viscous term $\mu\boldsymbol{\nabla}^2\mathbf{u}$, so it is necessary to impose more boundary conditions for a viscous fluid than for an inviscid fluid.

## 1.7 Boundary conditions

### 1.7.1 Boundary conditions at a rigid impermeable boundary

- Suppose the fluid is in contact with a rigid impermeable surface $S$ that has unit normal $\mathbf{n}$ pointing out of the fluid and velocity $\mathbf{U}$.
- Since fluid cannot flow though the impermeable surface, we prescribe the no-flux condition that

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{U}\cdot\mathbf{n} \text{ on } S,$$

so that the normal velocity components of the fluid and boundary are equal.

- For a viscous fluid, we also impose the no-slip condition

$$\mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = \mathbf{U} - (\mathbf{U}\cdot\mathbf{n})\mathbf{n} \text{ on } S,$$

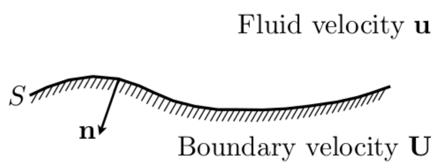
so that the tangential velocity components of the fluid and boundary are equal.

- Hence, the combined no-flux and no-slip boundary conditions are given by

$$\mathbf{u} = \mathbf{U} \text{ on } S,$$

so that the velocities of the fluid and boundary are equal.

- Note that we have prescribed a total of three scalar boundary conditions.



### 1.7.2 Boundary conditions at a free surface

- Suppose the fluid has a free boundary $\Gamma$ that has unit normal $\mathbf{n}$ pointing out of the fluid and outward normal velocity $V$, the free boundary separating the fluid from a vacuum and being unknown *a priori*.
- Assuming there is no evaporation, we prescribe the no-flux condition that

$$\mathbf{u} \cdot \mathbf{n} = V \text{ on } \Gamma,$$

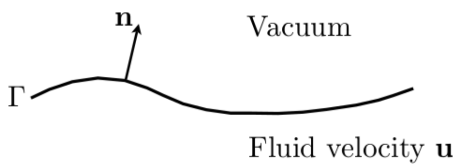
so that the normal velocity components of the fluid and free boundary are equal.

- Instead of the no-slip condition, we prescribe in the absence of surface tension the no-stress condition that

$$\mathbf{t}(\mathbf{n}) = 0 \text{ on } \Gamma,$$

since the vacuum exerts no surface traction on the fluid.

- Note that we have prescribed a total of four scalar boundary conditions, one more than for a rigid impermeable boundary because we need to prescribe an additional equation to determine the location of the free boundary.




**Remarks**:
For a free surface $y = f(x,t)$ the kinematic condition is

$$\text{on } y = f(x,t): \qquad \frac{D}{Dt}(y - f(x,t)) = 0,$$

representing that the fluid particles on the surface stay in the surface. Using the definition of the material derivative, we obtain

$$\text{on } y = f(x,t): \qquad \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = v, \tag{A}$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$. We remark that this is equivalent to the no-flux condition

$$\text{on } y = f(x,t): \qquad \mathbf{u} \cdot \mathbf{n} = V, \tag{B}$$

where $V$ is the outward normal velocity of the surface. This follows since the position vector for a location on the surface is $\mathbf{r} = (x, f(x,t))$ which gives the unit tangent and normal to the surface as

$$\hat{\mathbf{t}} = \frac{(1, f_x)}{(1+f_x^2)^{1/2}}, \qquad \mathbf{n} = \frac{(-f_x, 1)}{(1+f_x^2)^{1/2}}.$$

Consequently (A) can be written as

$$\mathbf{u} \cdot \mathbf{n} = \frac{f_t}{(1+f_x^2)^{1/2}}$$

where

$$V = \frac{d\mathbf{r}}{dt} \cdot \mathbf{n} = (\dot{x}, f_x\dot{x} + f_t) \cdot \frac{(-f_x, 1)}{(1+f_x^2)^{1/2}} = \frac{f_t}{(1+f_x^2)^{1/2}}$$

and thus (B) follows.

## 1.8 Vorticity

- Vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ is a measure of the local rotation of fluid elements.

- Assuming the body force is conservative (so that $\boldsymbol{\nabla} \wedge \mathbf{F} = 0$) and taking curl of the momentum equation (23), we obtain the vorticity transport equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} = \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega},$$

  where the kinematic viscosity $\nu = \mu/\rho$.

- The effect of viscosity is to diffuse vorticity.

- In two-dimensions with velocity

$$\mathbf{u} = u(x,\ y,\ t)\mathbf{i} + v(x,\ y,\ t)\mathbf{j},$$

  the vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u} = \omega(x,\ y,\ t)\mathbf{k},$$

  where the $z$-component of vorticity is given by

$$\omega \ = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

  Hence, $(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} = \omega \partial \mathbf{u}/\partial z = 0$, and we obtain the two-dimensional vorticity transport equation

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = \frac{\partial w}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \boldsymbol{\nabla}^2 \omega. \tag{24}$$

- For an inviscid fluid ($\nu = 0$) , the two-dimensional vorticity transport equation becomes

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0.$$

  Thus, if $\omega = 0$ at time $t = 0$, then $\omega = 0$ for all $t > 0$ (Cauchy-Lagrange Theorem).

- Since $\boldsymbol{\nabla}^2 \omega \equiv 0$ if $\omega \equiv 0$, might expect that adding diffusion ($\nu > 0$) doesn't change the argument. This is incorrect because vorticity is generated at boundaries.

- To see this, use the fact the flow is incompressible to write (24) in the conservative form

$$\frac{\partial \omega}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{Q} = 0,$$

  where the "vorticity flux" $\mathbf{Q} = \omega \mathbf{u} - \nu \boldsymbol{\nabla} \omega$; the two terms on the RHS of this expression correspond to convective and diffusive transport of vorticity.

- Consider a stationary rigid boundary $S$ with unit normal $\mathbf{n}$ pointing out of the fluid. In general the no-slip condition ($\mathbf{u} = 0$ on $S$) does not imply that $\mathbf{Q} \cdot \mathbf{n} = 0$ on $S$. In particular, the boundary acts as an effective source (sink) of vorticity if $\mathbf{Q} \cdot \mathbf{n} < 0$ ($\mathbf{Q} \cdot \mathbf{n} > 0$).

  **Remarks**:
  Compare to a convection-diffusion equation for concentration c (or temperature) of the form

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = D \nabla^2 c$$

  which can be put in conservative form

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{H} = 0, \tag{1.8.1}$$

where

$$\mathbf{H} = \mathbf{H}_{\text{convection}} + \mathbf{H}_{\text{diffusion}} = c\mathbf{u} - D\nabla c,$$

and for incompressible flow $\nabla \cdot (c\mathbf{u}) = c\underbrace{\nabla \cdot \mathbf{u}}_{=0} + \mathbf{u} \cdot \nabla c$. Note that integrating (1.8.1) over a fixed volume V with unit outward normal $\mathbf{n}$ to its surface $\partial V$ gives

$$\frac{d}{dt}\int_V c\,dV = \int_V \frac{\partial c}{\partial t}\,dV = \int_V -\nabla \cdot \mathbf{H}\,dV = -\int_{\partial V} \mathbf{H} \cdot \mathbf{n}\,dS,$$

which is the conservation of the amount of $c$ in integral form. This illustrates an alternative approach to section 1.3 in which we could consider arbitrary fixed volume $V$, the motion of the fluid then being taken into account through a convective flux term. For example, for mass conservation with fixed $V$:

$$\frac{d}{dt}\int_V \rho\,dV = -\int_{\partial V} \rho\mathbf{u} \cdot \mathbf{n}\,dS \implies \int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u})\,dV = 0.$$

## 1.9 Conservation of energy

- We now consider the transport of energy by a conducting viscous fluid in the absence of external energy sources (*e.g.* radiation, chemical reactions); *cf.* the heat conduction example in §1.5.2.
- Consider a material volume $V(t)$ whose boundary $\partial V(t)$ has outward unit normal $\mathbf{n}$.
- The total internal energy in $V(t)$ due to heat and kinetic energy is

$$E(t) = \iiint_{V(t)} \rho c_v T + \frac{1}{2}\rho\,|\mathbf{u}|^2\,\mathrm{d}V,$$

where $c_v$ is the specific heat and $T(\mathbf{x},\ t)$ the absolute temperature.

- Conservation of energy states that the time rate of change of the total internal energy increases due to
  - (i) conduction of heat into $V(t)$ through $\partial V(t)$ , with net rate

$$\iint_{\partial V(t)} \mathbf{q} \cdot (-\mathbf{n})\mathrm{d}S,$$

    where the heat flux vector $\mathbf{q} = -k\boldsymbol{\nabla} T$ according to Fourier's law, $k$ being the thermal conductivity;
  - (ii) work done by surface stresses $\mathbf{t}(\mathbf{n})$ on $\partial V(t)$ , with net rate

$$\iint_{\partial V(t)} \mathbf{t}(\mathbf{n}) \cdot \mathbf{u}\mathrm{d}S;$$

  - (iii) work done by body forces in $V(t)$ , with net rate

$$\iiint_{V(t)} \rho\mathbf{F} \cdot \mathbf{u}\mathrm{d}V.$$

- Note that Fourier's law may be written in the form

$$\mathbf{q} = -k\boldsymbol{\nabla} T = -k\mathbf{e}_j\frac{\partial T}{\partial x_j}$$

and that Cauchy's stress theorem (12) implies

$$\mathbf{t}(\mathbf{n}) \cdot \mathbf{u} = (\mathbf{e}_i\sigma_{ij}n_j) \cdot (\mathbf{e}_k u_k) = \delta_{ik}u_k\sigma_{ij}n_j = u_i\sigma_{ij}n_j.$$

- Hence, conservation of energy for the material volume $V(t)$ may be written in the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \iint_{\partial V(t)} \left( k\frac{\partial T}{\partial x_j} + u_i\sigma_{ij} \right) n_j \mathrm{d}S + \iiint_{V(t)} \rho \mathbf{F}\cdot\mathbf{u}\mathrm{d}V.$$

- Using the corollary to Reynolds' transport theorem (8) and the divergence theorem implies that

$$\iiint_{V(t)} \rho\frac{\mathrm{D}}{\mathrm{D}t}\left( c_v T + \frac{1}{2}|\mathbf{u}|^2 \right) - \frac{\partial}{\partial x_j}\left( k\frac{\partial T}{\partial x_j} + u_i\sigma_{ij} \right) - \rho\mathbf{F}\cdot\mathbf{u}\mathrm{d}V = 0.$$

- Since $V(t)$ is arbitrary, the integrand must be zero (if it is continuous), *i.e.*

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left( c_v T + \frac{1}{2}|\mathbf{u}|^2 \right) = \frac{\partial}{\partial x_j}\left( k\frac{\partial T}{\partial x_j} + u_i\sigma_{ij} \right) + \rho\mathbf{F}\cdot\mathbf{u}.$$

- Assuming that $c_v$ and $k$ are constants, and using Cauchy's momentum equation gives

$$\rho c_v \frac{\mathrm{D}T}{\mathrm{D}t} = k\boldsymbol{\nabla}^2 T + \Phi, \qquad \Phi = \sigma_{ij}\frac{\partial u_i}{\partial x_j}.$$

- Finally, substituting the constitutive law for an incompressible Newtonian fluid implies that the viscous dissipation is given by

$$\Phi = \frac{1}{2}\mu\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2$$

- Hence, fluid deformation ($\Rightarrow \Phi > 0$) always increases the temperature.

### 1.10 Unidirectional flows

- There are hardly any explicit solutions of the Navier-Stokes equations. Almost all of them are for unidirectional flows in which there is one flow direction only.

- Choose $x$-axis in direction of flow, *i.e.* set $\mathbf{u} = u(x,\ y,\ z,\ t)\mathbf{i}$.

- Since

$$\mathbf{u}\cdot\boldsymbol{\nabla} = u\frac{\partial}{\partial x},$$

(22)–(23) become

$$\begin{aligned} \frac{\partial u}{\partial x} &= 0, \\ \rho\left( \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \right) &= -\frac{\partial p}{\partial x} + \mu\left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\ 0 &= -\frac{\partial p}{\partial y}, \\ 0 &= -\frac{\partial p}{\partial z}. \end{aligned}$$

- Hence, $u$ is independent of $x$; $p$ is independent of $y$ and $z$. It follows that $u = u(y,\ z,\ t)$ and $p = (x,\ t)$ satisfy

$$\rho\frac{\partial u}{\partial t} - \mu\left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = -\frac{\partial p}{\partial x}.$$

- LHS independent of $x$; RHS independent of $y$ and $z$; hence LHS = RHS independent of $x,\ y,\ z$.

- Hence, the pressure gradient is a function of time $t$ only, say

$$\frac{\partial p}{\partial x} = G(t) \ ,$$

which must be prescribed to find $u(y,\ z,\ t)$ .

- The $x$-component of velocity $u(y,\ z,\ t)$ satisfies the two-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \nu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{G(t)}{\rho}, \tag{25}$$

the kinematic viscosity $\nu = \mu/\rho$ being the diffusion coefficient and $G(t)$ being the applied pressure gradient.

**Remarks**

(i) In unidirectional flows, all nonlinear terms in the Navier-Stokes equations vanish: the convective term

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = 0.$$

(ii) The remaining equation (25) is linear and may be solved analytically using standard techniques in several physically relevant geometries (which must be invariant to translations in the $x$-direction, as the flow is in the $x$-direction).

(iii) In practical applications, the applied pressure gradient $G(t)$ is zero, constant or oscillatory.

(iv) Further simplifications:

(a) in one-dimensional steady unidirectional flow with $u = u(y)$ , (25) reduces to the ordinary differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{G}{\mu};$$

(b) in one-dimensional unsteady unidirectional flow with $u = u(y,\ t)$ , (25) reduces to the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{G(t)}{\rho};$$

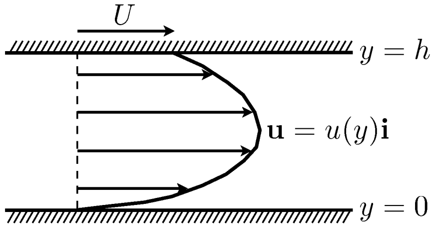
(c) in two-dimensional steady unidirectional flow with $u = u(y,\ z)$, (25) reduces to Poisson's equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{G}{\mu}.$$

(v) The partial differential equations in (b) and (c) are amenable to standard methods *e.g.* separation of variables and Fourier series methods, simularity reduction to an ordinary differential equation, integral transforms.

### 1.10.1 Example: Poiseuille/Couette flow in a channel

- Consider flow in a channel $0 < y < h$:



- Suppose lower plate at rest, upper plate moves to right with speed $U$ and constant applied pressure gradient

$$\frac{\partial p}{\partial x} = G.$$

- Assuming one-dimensional steady unidirectional flow with velocity $\mathbf{u} = u(y)\mathbf{i}$, the Navier-Stokes equations reduce to the ordinary differential equation

$$\mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = G,$$

which represents a balance of viscous shear forces and the applied pressure gradient.

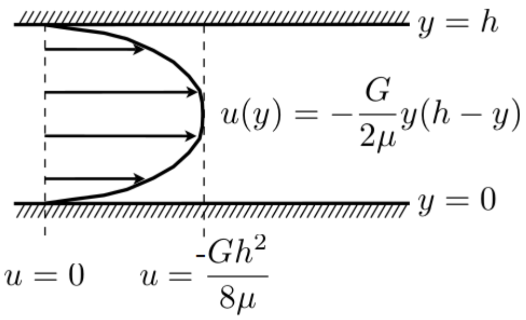
- The no-flux boundary conditions on the plates are satisfied automatically, while the no-slip boundary conditions imply that $u(0) = 0,\ u(h) = U$.

- Hence,

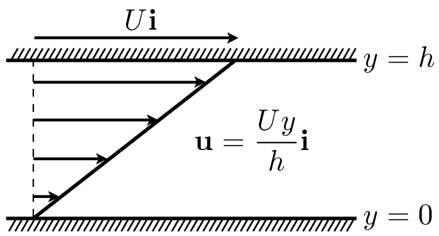
$$u(y) = -\frac{G}{2\mu} y(h-y) + \frac{Uy}{h}.$$

- Special cases:

  (i) **Poiseuille flow** ($U = 0$) driven by pressure gradient $G \neq 0$ has a quadratic velocity profile:

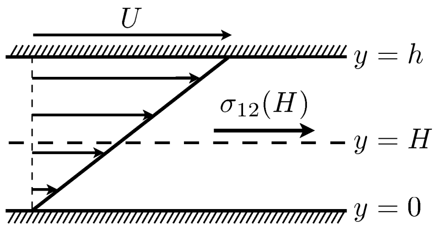



  (ii) **Couette flow** ($G = 0$) driven by moving plate has a linear velocity profile:




### 1.10.2 Example: Shear stress in a Couette flow

- Consider the Couette flow $\mathbf{u} = u(y)\mathbf{i}$, with $u(y) = \dfrac{Uy}{h}$:




- Flow occurs in layers $y = H$ (constant).

- Fluid above $y = H$ exerts a shear stress on fluid below $y = H$ (and vice versa).

- The shear stress is given by

$$\sigma_{12}(H) = \mu \frac{\mathrm{d}u}{\mathrm{d}y}(H) = \frac{\mu U}{h},$$

where the subscript 1 indicates the $x$-component of stress and the subscript 2 that the normal to $y = H$ is in the $y$-direction.

- This shear stress arises because fluid above $y = H$ is moving at a different speed than fluid below.

- Note that $\sigma_{12}(H) > 0$ for $U > 0$, as the fluid above $y = H$ is moving faster than fluid below, *i.e.* viscosity causes fluid above $y = H$ to "drag along" fluid below (*cf.* inviscid fluid $\mu = 0$).

- The shear stress exerted by the fluid on the lower plate is given by

$$\sigma_{12}(0) = \frac{\mu U}{h};$$

by Newton's third law, the shear stress exerted by the fluid on the upper plate is given by

$$-\sigma_{12}(H) = -\frac{\mu U}{h}.$$

- The force per unit area in the $x$-direction required to sustain the motion of the upper plate is

$$\sigma_{12}(H) = \frac{\mu U}{h} > 0 \text{ for } U > 0.$$

- If we can measure $\sigma_{12}(H)$ , $U$ and $h$ in our Couette flow rheometer, then we can calculate the viscosity $\mu$ of an incompressible Newtonian fluid.

### 1.10.3 Example: Flow down an inclined plane under gravity

- Let the plane make an angle $\alpha$ to the horizontal.
- Consider steady unidirectional 2-D flow: $\mathbf{u} = u(y)\mathbf{i} = (u(y), 0)$
- $\mathbf{g} = (g \sin\alpha, -g\cos\alpha)$

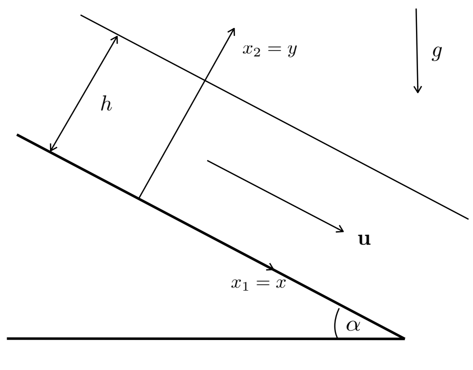



Figure 4: Flow down an inclined plane.

- The continuity equation is automatically satisfied for this flow field and the components of the momentum equation are:

$$\mathbf{i} \text{ direction:} \qquad 0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} + \rho g \sin\alpha \tag{10.1}$$

$$\mathbf{j} \text{ direction:} \qquad 0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha \tag{10.2}$$

- Introduce
$$G = \frac{\partial p}{\partial x} - \rho g \sin\alpha, \tag{10.3}$$
which from (10.1) is a constant and becomes
$$\mu \frac{d^2 u}{dy^2} = G \implies u(y) = \frac{G}{2\mu} y^2 + Ay + B$$
for arbitrary constants $A, B$.

- The inclined plane $y = 0$ is assumed fixed, the no-slip condition gives
$$u(0) = 0 \implies B = 0.$$

- The upper surface $y = h$ is assumed level and the stress balances the external (constant) air pressure $p_a$:
$$\text{On } y = h: \qquad \mathbf{t} = -p_a \mathbf{n} \qquad \mathbf{n} = \mathbf{j} = (0, 1).$$

Using $t_i = \sigma_{ij} n_j = \sigma_{i2}$ we have
$$\text{On } y = h: \qquad \sigma_{12} = 0, \quad \sigma_{22} = -p_a.$$
However $\sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ with $u_1 = u(y), u_2 = 0$ and $x_1 = x, x_2 = y$. Thus $\sigma_{12} = \mu \frac{du}{dy}, \sigma_{22} = -p$ so that
$$\text{On } y = h: \qquad \underbrace{\mu \frac{du}{dy} = 0}_{\text{shear stress}}, \qquad \underbrace{p = p_a}_{\text{normal stress}}.$$
Thus $A = -\frac{Gh}{\mu}$ and hence
$$u = -\frac{G}{2\mu} y(2h - y).$$

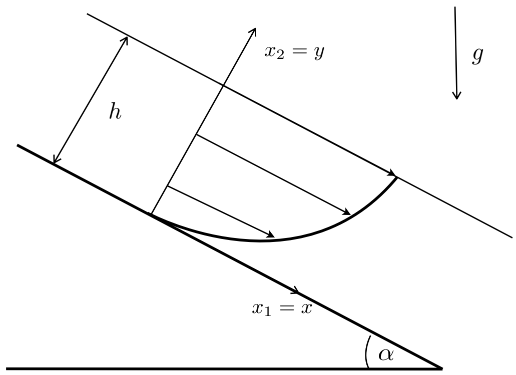



Figure 5: Flow profile down an inclined plane.

- The pressure can be calculated from (10.2) and (10.3) as
$$p = Gx + \rho g x \sin\alpha - \rho g y \cos\alpha + C$$
for an arbitrary constant $C$. Using the normal stress condition on the free surface gives
$$p_a = x(G + \rho g \sin\alpha) + C - \rho g h \cos\alpha,$$
which holds for all $x$ and hence
$$G = -\rho g \sin\alpha, \qquad C = p_a + \rho g h \cos\alpha.$$
Thus $\qquad p = p_a + \rho g (h - y) \cos\alpha.$

- Note: Both $u, p$ are independent of $x$, which is consistent with the problem being translationally invariant wrt $x$.

- The volume flux (per unit length in the z-direction) is
$$Q = \int_0^h u \, dy = \frac{\rho g h^3 \sin\alpha}{3\mu},$$
which is proportional to the cube of the depth $h$.

[The volume flux is defined as the volume of fluid crossing a cross-section per unit time:
$$Q = \int_S \mathbf{u} \cdot \mathbf{n} \, dS = \int_{z=0}^{L_3} \int_{y=0}^{h} u \, dy = L_3 \int_0^h u \, dy,$$
where $\mathbf{n}$ is a unit normal to the cross-section in the direction of flow. ]

### 1.10.4 Flows with circular streamlines

- In cylindrical coordinates $(r, \theta, z)$ with velocity $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$, the Navier-Stokes equations take the form

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{10.4}$$

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \tag{10.5}$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right] \tag{10.6}$$

$$\frac{\partial u_z}{\partial t} + (\mathbf{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z \tag{10.7}$$

where

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z}, \qquad \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}.$$

- For planar circular flow

$$\mathbf{u} = u_\theta(r,t)\,\mathbf{e}_\theta \qquad \text{with } u_r = u_z = 0, \tag{10.8}$$

for which the streamlines are circular.

[ $\mathbf{u} = \nabla\wedge(\psi\mathbf{e}_z) = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\mathbf{e}_r - \frac{\partial \psi}{\partial r}\mathbf{e}_\theta \implies \frac{\partial \psi}{\partial \theta} = 0 \implies \psi = \psi(r,t)$ which give streamlines $\psi$ =constant as circles $r$ =constant at a given time $t$. ]

- The circular flow field (10.8) automatically satisfies (10.4), whilst (10.5)–(10.7) become

$$-\rho\frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} \tag{10.9}$$

$$\rho\frac{\partial u_\theta}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) \tag{10.10}$$

$$0 = -\frac{\partial p}{\partial z} \tag{10.11}$$

(10.11) $\implies p = p(r,\theta,t)$

(10.10) $\implies \frac{\partial p}{\partial \theta} = P_0(r,t) \implies p = P_0(r,t)\theta + P_1(r,t)$ for some functions $P_0, P_1$. But $p$ should be a single valued function of position and not multivalued $\implies P_0 = 0$. Hence $p = p(r,t)$ and we obtain

$$\rho\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}, \qquad \rho\frac{\partial u_\theta}{\partial t} = \mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right). \tag{10.12}$$

- **Example: Steady flow between rotating cylinders**

Consider two circular cylinders rotating about a common axis with angular speeds $\Omega_1$ and $\Omega_2$ at distances $R_1$ and $R_2$ respectively. For steady flow (10.12) gives

$$r^2\frac{d^2u_\theta}{dr^2} + r\frac{du_\theta}{dr} - u_\theta = 0 \qquad R_1 < r < R_2. \tag{10.13}$$

No-slip conditions on the cylinders are

$$\text{at } r = R_1\text{:} \quad u_\theta = \Omega_1 R_1, \qquad \text{at } r = R_2\text{:} \quad u_\theta = \Omega_2 R_2. \tag{10.14}$$

The equation (10.13) is an Euler equation and is equidimensional in $r$ (i.e. scale invariant under the transformation $r = \alpha\bar{r}, u_\theta = \bar{u}_\theta$ for any $\alpha$). Let $z = \ln r$ then

$$\frac{d}{dr} = \frac{d}{dz}\frac{dz}{dr} = \frac{1}{r}\frac{d}{dz} \implies r\frac{d}{dr} = \frac{d}{dz}$$

and

$$r\frac{d}{dr}\left(r\frac{d}{dr}\right) = \frac{d^2}{dz^2} \quad \text{or} \quad r^2\frac{d^2}{dr^2} + r\frac{d}{dr} = \frac{d^2}{dz^2}$$

Thus (10.13) becomes

$$\frac{d^2u_\theta}{dz^2} - u_\theta = 0 \implies u_\theta = Ae^z + Be^{-z} = Ar + \frac{B}{r}.$$

The conditions (10.14) determine the constants as

$$A = \frac{\Omega_2R_2^2 - \Omega_1R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{(\Omega_1 - \Omega_2)R_1^2R_2^2}{R_2^2 - R_1^2}.$$

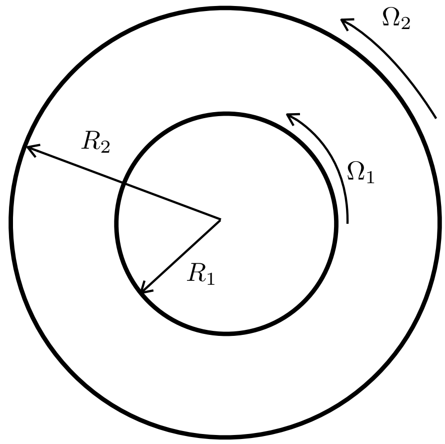
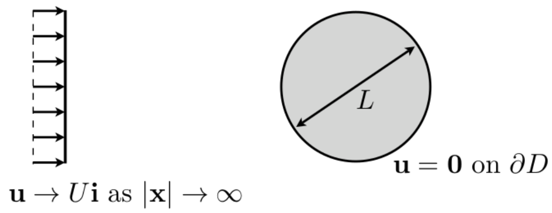



Figure 6: Flow between concentric cylinders.

## 1.11 Dimensionless Navier-Stokes equations

- As with all mathematical modelling, we will only ever start to understand the mathematical implications if we write the model in dimensionless variables.
- Consider the flow of an incompressible Newtonian fluid with far-field velocity $U\mathbf{i}$ past a stationary obstacle of typical size $L$ and with boundary $\partial D$.




- In the absence of body forces, the flow is governed by the incompressible Navier-Stokes equations (22)-(23) with $\mathbf{F} = 0$, *i.e.*

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right) = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\mathbf{u}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0,$$

  with boundary conditions in the diagram above.
- Here and hereafter we will denote by $[x]$ the typical dimensional size of the quantity $x$.
- The typical dimensional sizes of the dependent and independent variables are given by

$$\begin{aligned} \text{length scale:}[\mathbf{x}] &= L; \\ \text{velocity scale:}[\mathbf{u}] &= U; \\ \text{time scale:}[t] &= \frac{L}{U}; \\ \text{pressure scale:}[p] &= \text{ to be determined.} \end{aligned}$$

- Hence, we nondimensionalize by scaling

$$\mathbf{x} = L\hat{\mathbf{x}}, \quad \mathbf{u} = U\hat{\mathbf{u}}, \quad t = \frac{L}{U}\hat{t}, \quad p = p_{atm} + [p]\hat{p}.$$

  where $p_{atm}$ is the atmospheric pressure.

- Since $x_i = L\hat{x}_i$,

$$\boldsymbol{\nabla} = \mathbf{e}_i \frac{\partial}{\partial x_i} = \frac{1}{L}\mathbf{e}_i \frac{\partial}{\partial \hat{x}_i} = \frac{1}{L}\hat{\boldsymbol{\nabla}}.$$

- The incompressibility condition $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ becomes

$$\frac{1}{L}\hat{\boldsymbol{\nabla}} \cdot (U\hat{\mathbf{u}}) = 0 \quad \Rightarrow \quad \hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}} = 0.$$

- Similarly, the momentum equation becomes

$$\frac{\rho U^2}{L}\left(\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \hat{\boldsymbol{\nabla}})\hat{\mathbf{u}}\right) = -\frac{[p]}{L}\hat{\boldsymbol{\nabla}}\hat{p} + \frac{\mu U}{L^2}\hat{\boldsymbol{\nabla}}^2\hat{\mathbf{u}}.$$

- The ratio of the inertia term on the LHS to the viscous term on the RHS is given by

$$\frac{[\text{inertia term}]}{[\text{viscous term}]} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{\rho L U}{\mu} = \frac{LU}{\nu} = Re,$$

which is the (dimensionless) Reynolds number.

- Note that two flows are dynamically similar if they satisfy the same dimensionless problem (*i.e.* same geometry, governing equations, boundary conditions and dimensionless parameters).

- We will study both high and low Reynolds number flows.

**High-Reynolds number flows** $Re \gg 1$

- Choose the inviscid pressure scale $[p] = \rho U^2$ to obtain

$$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \hat{\boldsymbol{\nabla}})\,\hat{\mathbf{u}} = -\hat{\boldsymbol{\nabla}}\hat{p} + \frac{1}{Re}\hat{\boldsymbol{\nabla}}^2\hat{\mathbf{u}} \quad \hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}} = 0.$$

- In this regime we hope to ignore the small viscous term and solve the inviscid Euler equations except in thin layers on boundaries where viscosity is required to satisfy the no-slip boundary condition.

**Low-Reynolds number flows** $Re \ll 1$

- Choose the viscous pressure scale $[p] = \mu U/L$ to obtain

$$Re\left(\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \hat{\boldsymbol{\nabla}})\hat{\mathbf{u}}\right) = -\hat{\boldsymbol{\nabla}}\hat{p} + \hat{\boldsymbol{\nabla}}^2\hat{\mathbf{u}} \quad \hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}} = 0.$$

- In this regime we hope to ignore the inertia term and solve the resulting slow-flow equations:

$$\hat{\boldsymbol{\nabla}}^2\hat{\mathbf{u}} = \hat{\boldsymbol{\nabla}}\hat{p}, \quad \hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}} = 0.$$

**When is viscosity important in practice?**

- Typical values of $L$, $U$, $\nu$ and hence $Re = LU/\nu$ for a car travelling at 30mph through air, a fish swimming in water and for a marble falling through treacle are shown in the following table.

| Object | $L$ | $U$ | $\nu$ | $Re$ |
|---|---|---|---|---|
| Car | 1 m | $10\text{ms}^{-1}$ | $10^{-5}\text{m}^2\text{s}^{-1}$ | $10^6$ |
| Fish | 0.1 m | $0.1\text{ms}^{-1}$ | $10^{-6}\text{m}^2\text{s}^{-1}$ | $10^4$ |
| Marble | 1cm | $1\text{cms}^{-1}$ | $10^3\text{cm}^2\text{s}^{-1}$ | $10^{-3}$ |

**Remarks**

- The Reynolds number is large for many everyday flows.
- Warning: Solution may generate its own length scale (*e.g.* tornado).
- If the Reynolds number is of order unity, then the Navier-Stokes equations must be solved numerically. However, modern computers can't get much past $Re = 10^4$ in realistic geometries.
- We will consider both large and small Reynolds number flows using asymptotics.

**Nondimensionalisation**

Nondimensionalisation is an important first step in the analysis of a system of equations for the following reasons:

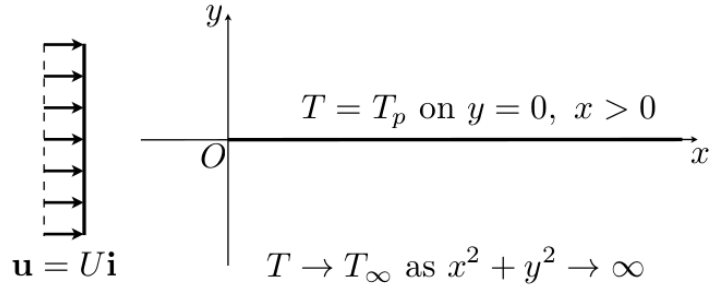
- It identifies the dimensionless groups which control the solution behaviour. Further the sizes ofthe dimensionaless groups determines the extent and influence on the solution. As such, the solution behaviour may be characterized in terms of such groups.
- It provides information on the dimensional parameters necessary if the physical process is to be simulated/recreated experimentally on different scales. An example would be simulating flight in a wind tunnel, where the Re number should be kept the same as in the physical situation. If a plane has typical wing span of $\mathrm{L} = 30\mathrm{m}$ and travels at $\mathbf{u} = 500\mathrm{mph} = 223\mathrm{m/s}$ then for the model in the wind tunnel of typical length $\mathrm{L}_1$, the air needs to be driven at a speed of $\mathbf{u}_1 = \mathbf{u}\mathrm{L}/\mathrm{L}_1$ for the Re number of the two situations to be the same.
- Finally, it usually reduces the number of parameters in the model.

# 2 High Reynolds number flows

## 2.1 Thermal boundary layer on a semi-infinite flat plate

### 2.1.1 Dimensional problem

- Paradigm for viscous boundary layer on a semi-infinite flat plate.
- The two-dimensional steady heat convection-conduction problem consists of
  - inviscid fluid, velocity $U\mathbf{i}$, temperature $T_\infty$ upstream;
  - plate at $y = 0,\ x > 0$, held at temperature $T_p$.




- Energy equation for temperature $T$, with $\mu = 0$:

$$\rho c_v \left( \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) T \right) = k \boldsymbol{\nabla}^2 T.$$

- We seek a steady solution $T = T(x,\ y)$ , with $\mathbf{u} = U\mathbf{i}$, so that

$$U \frac{\partial T}{\partial x} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) ,$$

  where $\kappa = k/\rho c_v$ is the thermal diffusivity (units $\mathrm{m}^2\mathrm{s}^{-1}$).

### 2.1.2 Dimensionless problem

- Choose arbitrary length scale $L$ and set

$$x = L\hat{x},\ y = L\hat{y},\ T = T_\infty + (T_p - T_\infty)\hat{T}.$$

- The energy equation becomes

$$Pe \frac{\partial \hat{T}}{\partial \hat{x}} = \frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{y}^2},$$

  where the Peclet number $Pe = LU/\kappa$; *cf.* $Re = LU/\nu$.

- Boundary condition on plate: $\hat{T} = 1$ on $\hat{y} = 0 < \hat{x}$.
- Boundary condition at infinity: $\hat{T} \to 0$ as $\hat{x}^2 + \hat{y}^2 \to \infty$.
- The time scale for heat energy to convect a distance $L$ is $L/U$, while the time scale for heat energy to diffuse a distance $L$ is $L^2/\kappa$. Hence, the Peclet number

$$Pe = \frac{L^2/\kappa}{L/U} = \frac{\text{diffusion timescale}}{\text{convection timescale}}.$$

- We seek a solution for $Pe \gg 1$, *i.e.* $\epsilon = 1/Pe \ll 1$.

- Dropping hats on the dimensionless variables, the dimensionless problem is given by

$$\frac{\partial T}{\partial x} = \varepsilon\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{26}$$

where $\varepsilon = 1/Pe \ll 1$, with boundary conditions

$$T = 1 \text{ on } y = 0,\ x > 0 \tag{27}$$

and

$$T \to 0 \text{ as } x^2 + y^2 \to \infty. \tag{28}$$

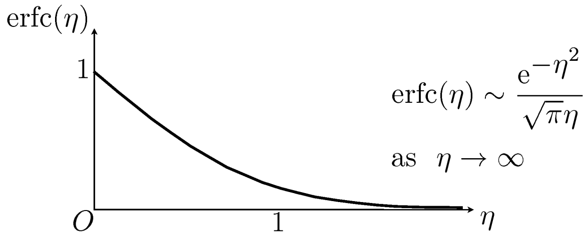
### 2.1.3 Exact solution

- The exact solution to (26)-(28) is given by

$$T(x,\ y) = \mathrm{erfc}(\eta) \equiv \frac{2}{\sqrt{\pi}}\int_\eta^\infty \mathrm{e}^{-s^2}\mathrm{d}s,$$

where erfc is the complementary error function and

$$\eta(x,\ y) = \left[\frac{(x^2+y^2)^{1/2} - x}{2\varepsilon}\right]^{1/2}$$




- Key observation: $y^2 = \varepsilon(4\eta^2 x) + \varepsilon^2(4\eta^4)$ .

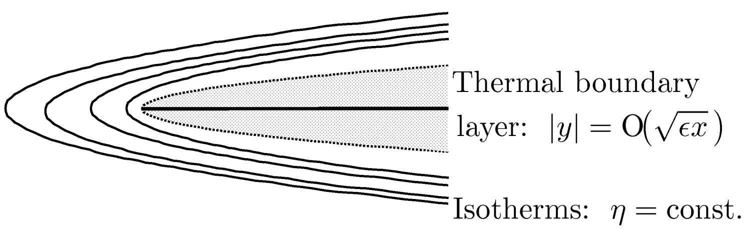
**Deductions from $T = \mathrm{erfc}(\eta)$, $y^2 = \varepsilon(4\eta^2 x) + \varepsilon^2(4\eta^4)$**

(i) Isotherms $\eta =$ constant are parabolae.

(ii) For T not close to zero we need $\eta = \mathrm{O}(1)$ as $\varepsilon \to 0$. As $\varepsilon \to 0$ with $x = \mathrm{O}(1)$ ,

$$\eta = \mathrm{O}(1) \quad \Rightarrow \quad y^2 \sim \varepsilon(4\eta^2 x) \quad \Rightarrow \quad \eta \sim \frac{|y|}{\sqrt{4\varepsilon x}} \quad \Rightarrow \quad T \sim \mathrm{erfc}\left(\frac{|y|}{\sqrt{4\varepsilon x}}\right). \tag{29}$$

Hence, there is a thermal boundary layer on the plate in which $|y| = \mathrm{O}(\sqrt{\varepsilon x})$ as $\varepsilon \to 0$ with $x = \mathrm{O}(1)$ , as illustrated.



### 2.1.4 Boundary layer analysis

- Instead of solving exactly and then expanding, let us expand first and then solve.
- In outer region away from plate, expand

$$T \sim \overline{T}_0 + \varepsilon \overline{T}_1 + \cdots \qquad \Rightarrow \qquad \frac{\partial \overline{T}_0}{\partial x} = 0 \qquad \Rightarrow \qquad \overline{T}_0 = 0,$$

  by the upstream boundary condition (28); in fact, $T = \mathrm{O}(\varepsilon^n)$ as $\varepsilon \to 0$ for all integer $n$, as we know from the exact solution that $T$ is exponentially small as $\varepsilon \to 0$ with $|y| = \mathrm{O}(1)$ .
- To determine the thickness of the thermal boundary layer $\delta = \delta(\varepsilon)$ on the plate as $\varepsilon \to 0$, we scale $y = \delta Y$ so that (26) becomes

$$\frac{\partial T}{\partial x} = \varepsilon \frac{\partial^2 T}{\partial x^2} + \frac{\varepsilon}{\delta^2} \frac{\partial^2 T}{\partial Y^2}$$

- Since the LHS is of $\mathrm{O}(1)$ , while the RHS is of $\mathrm{O}(\varepsilon/\delta^2)$ , it is necessary that $\varepsilon/\delta^2 = \mathrm{O}(1)$ for a nontrivial balance involving both convection and diffusion of heat energy.
- Hence, we set (without loss of generality) $\delta = \varepsilon^{1/2}$, so that $y = \varepsilon^{1/2} Y$ and

$$\frac{\partial T}{\partial x} = \varepsilon \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial Y^2}.$$

- Now expand $T \sim T_0(x,\ Y) + \varepsilon T_1(x,\ Y) + \cdots$ to obtain the leading-order thermal boundary layer equation

$$\frac{\partial T_0}{\partial x} = \frac{\partial^2 T_0}{\partial Y^2}. \tag{30}$$

- This is the heat equation with $x$ playing the role of time.
- The boundary condition on the plate (27) becomes

$$T_0 = 1 \text{ on } Y = 0 < x. \tag{31}$$

- To ensure that the boundary layer and outer expansions match (*i.e.* that they coincide in some intermediate overlap region), we impose the matching condition

$$T_0(x,\ Y) \to 0 \text{ as } |Y| \to \infty. \tag{32}$$

- The leading-order thermal-boundary-layer problem (30)-(32) has a similarity solution, *viz.*

$$T_0(x,\ Y) = \operatorname{erfc}\left(\frac{|Y|}{\sqrt{4x}}\right),$$

  which by (29) is the leading-order term in the expansion of the exact solution as $\varepsilon \to 0$ because $Y = y/\varepsilon^{1/2}$.

**Further remarks**:

<u>Outer solution</u>:
In $x = O(1), y = O(1)$ we pose

$$T = \bar{T}_0(x, y) + \epsilon \bar{T}_1(x, y) + \dots \qquad \text{as } \epsilon \to 0$$

to obtain
At $O(\epsilon^0)$: $\dfrac{\partial \bar{T}_0}{\partial x} = 0 \implies \bar{T}_0 = \bar{T}_0(y) = 0$ using the far-field condition that $\bar{T}_0 = 0$ as $x^2 + y^2 \to \infty$.
At $O(\epsilon^1)$: $\dfrac{\partial \bar{T}_1}{\partial x} = 0 \implies \bar{T}_1 = \bar{T}_1(y) = 0$ using the far-field condition that $\bar{T}_1 = 0$ as $x^2 + y^2 \to \infty$.
Continuing in this manner, it may be shown that the outer solution is $T = 0$ to all algebraic powers of $\epsilon$. However this solution doesn't satisfy the plate condition $T = 1$, suggesting the presence of a boundary layer or inner region near the plate.

<u>Inner solution</u>:

We scale as follows: $x = x, y = \delta Y, T = T$ where $\delta = \delta(\epsilon) \ll 1$. In $x = O(1), Y = O(1)$ we obtain

$$\frac{\partial T}{\partial x} = \frac{\epsilon}{\delta^2}\frac{\partial^2 T}{\partial Y^2} + \epsilon\frac{\partial^2 T}{\partial x^2}.$$

Dominant balance is given by $\delta = \epsilon^{1/2}$, so that

$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial Y^2} + \epsilon\frac{\partial^2 T}{\partial x^2}, \tag{2.1}$$

with the plate condition

$$\text{at } Y = 0, x > 0: \qquad T = 1,$$

and the matching condition with the outer

$$\text{as } Y \to \infty, x > 0: \qquad T \to 0,$$

which may also be referred to as the far-field condition for the boundary layer. Posing

$$T = T_0(x, Y) + \epsilon T_1(x, Y) + \dots \qquad \text{as } \epsilon \to 0$$

we obtain at $O(\epsilon^0)$:

$$\frac{\partial T_0}{\partial x} = \frac{\partial^2 T_0}{\partial Y^2}, \tag{30}$$

$$\text{on } Y = 0, x > 0: \qquad T_0 = 1, \tag{31}$$

$$\text{as } Y \to \infty, x > 0: \qquad T_0 \to 0. \tag{32}$$

This problem is scale invariant under the group scaling

$$x = \alpha\hat{x}, \quad Y = \beta\hat{Y}, \quad T_0 = \gamma\hat{T}_0,$$

with $\beta = \alpha^{1/2}$ for invariance of (30) and $\gamma = 1$ for invariance of (31). [Invariance means that the problem in hat variables is the same as the original unhat problem.] Hence

$$T_0(x, Y) = \hat{T}_0(\hat{x}, \hat{Y}) = \hat{T}_0\left(\frac{x}{\alpha}, \frac{Y}{\alpha^{1/2}}\right) = \hat{T}_0\left(1, \frac{Y}{x^{1/2}}\right)$$

choosing $\alpha = x$ since the scaling group holds for $\alpha \in \mathbb{R}_+$, derives seeking a similarity solution in the form

$$T_0(x, Y) = f(\eta), \qquad \eta = \frac{Y}{2\sqrt{x}}.$$

Performing the partial derivatives:

$$\frac{\partial T_0}{\partial x} = f'(\eta)\frac{\partial \eta}{\partial x} = -\frac{Y}{4x^{3/2}}f' = -\frac{\eta}{2x}f', \quad \frac{\partial T_0}{\partial Y} = f'(\eta)\frac{\partial \eta}{\partial Y} = \frac{1}{2x^{1/2}}f', \quad \frac{\partial^2 T_0}{\partial Y^2} = \frac{1}{2x^{1/2}}f''(\eta)\frac{\partial \eta}{\partial Y} = \frac{1}{4x}f'',$$

we obtain the two-point boundary value problem:

$$f'' + 2\eta f' = 0, \qquad \text{with } f(0) = 1, \quad f(\infty) = 0, \tag{2.2}$$

which has the solution

$$f(\eta) = \operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-s^2} ds.$$

We remark that alternatively we could seek a similarity solution in the form

$$T_0 = f(\eta), \qquad \eta = \frac{1}{2}Y x^{-n},$$

for some $n$ to be determined. Then

$$\frac{\partial T_0}{\partial x} = f'(\eta)\frac{\partial \eta}{\partial x} = -\frac{nY}{2x^{n+1}}f' = -\frac{n\eta}{x}f', \quad \frac{\partial T_0}{\partial Y} = f'\frac{\partial \eta}{\partial Y} = \frac{1}{2x^n}f', \quad \frac{\partial^2 T_0}{\partial Y^2} = \frac{1}{4x^{2n}}f'',$$

and (30) gives

$$f'' + 4nx^{2n-1}\eta f' = 0$$

which must be independent of $x$ (since $f$ is a function of $\eta$ only). Hence $n = 1/2$ and we recover the above two-point boundary value problem (2.2) for $f$, when the boundary conditions are taken as well.

Note: The dominant balance in (2.1) changes when $x = O(\epsilon)$ i.e. near to the plate edge. In $x = O(\epsilon), y = O(\epsilon)$ we then recover the full equation (i.e. all terms in (2.1)).

#### 2.1.5 Conclusions

- For $\varepsilon = 0$ (no diffusion) the upstream condition demands that $T \equiv 0$, which doesn't satisfy the boundary condition on the plate.
- For $0 < \varepsilon \ll 1$, this solution applies at leading order except in a thin boundary layer near the plate in which thermal diffusion (via the $\varepsilon\nabla^2 T$ term) increases the temperature from its upstream value to that on the plate.
- This is a singular perturbation problem as a uniformly valid approximation cannot be obtained by setting the small parameter equal to zero (cf. examples of regular and singular perturbation problems
- Singular behaviour arises because the small parameter $\varepsilon$ multiplies the highest derivative in (26).
- The highest derivative can be ignored except in thin regions where it is sufficiently large that it is no longer annihilated by the premultiplying small parameter.
- Such regions usually occur near the boundary of the domain, and so they are called boundary layers.

### 2.2 Viscous boundary layer on a semi-infinite flat plate

#### 2.2.1 Dimensional problem

- Consider the two-dimensional steady incompressible viscous flow of a uniform stream $U\mathbf{i}$ past a semi-infinite flat plate at $y = 0 < x$.
- In the absence of body forces, the flow is governed by the incompressible Navier-Stokes equations (22)-(23) with $\mathrm{F} = 0$, which become

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (33)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \quad (34)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (35)$$

where $\mathbf{u} = u(x,\ y)\mathbf{i} + v(x,\ y)\mathbf{j}$ is the velocity, $p(x,\ y)$ is the pressure, $\rho$ is the constant density and $\mu$ is the constant viscosity.

- The no-flux and no-slip boundary conditions on the plate are given by

$$u = 0,\ v = 0 \text{ on } y = 0,\ x > 0. \quad (36)$$

- The far-field boundary conditions are given by

$$u \to U,\ v \to 0 \text{ as } x^2 + y^2 \to \infty. \quad (37)$$

### 2.2.2 Dimensionless problem

- Choose arbitrary length scale $L$ and set

$$\mathbf{x} = L\hat{\mathbf{x}}, \ \mathbf{u} = U\hat{\mathbf{u}}, \ p = \rho U^2 \hat{p}$$

to obtain (dropping the hats $\hat{}$):

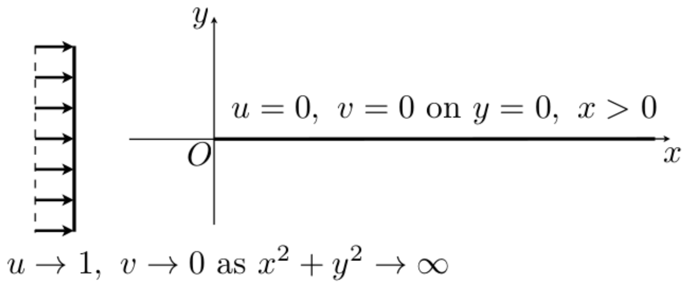
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{38}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{39}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{40}$$

where the Reynolds number $Re = LU/\nu$.

- The no-flux and no-slip boundary conditions on the plate (36) are unchanged, while the far-field conditions (37) pertain with $U = 1$, as illustrated in the following diagram.




- By (40), there exists a streamfunction $\psi(x, \ y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

- Eliminating $p$ between (38) and (39) by taking

$$\frac{\partial}{\partial y}(38) \ - \frac{\partial}{\partial x}(39)$$

and substituting for $u$ and $v$ gives

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\boldsymbol{\nabla}^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\boldsymbol{\nabla}^2\psi = \frac{1}{Re}\boldsymbol{\nabla}^2(\boldsymbol{\nabla}^2\psi) \equiv \frac{1}{Re}\boldsymbol{\nabla}^4\psi,$$

where $\boldsymbol{\nabla}^4$ is the biharmonic operator.

- Setting $\varepsilon = 1/Re$, we have

$$\frac{\partial(\psi, \boldsymbol{\nabla}^2\psi)}{\partial(y, x)} = \varepsilon\boldsymbol{\nabla}^4\psi. \tag{41}$$

- The boundary conditions on plate become

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \text{ on } y = 0, \ x > 0,$$

so we may set

$$\psi = \frac{\partial \psi}{\partial y} = 0 \text{ on } y = 0, \ x > 0 \tag{42}$$

without loss of generality; that $\psi$ is constant on the plate means that it is a streamline.

- The far-field boundary conditions imply that

$$\psi \sim y \text{ as } x^2 + y^2 \to \infty. \tag{43}$$

### 2.2.3 High Reynolds number regime

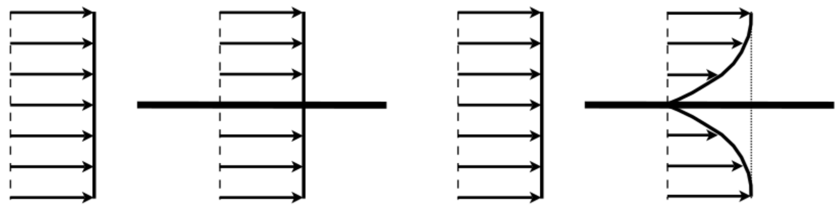
- We seek a solution to (41)–(43) for $Re \gg 1$, *i.e.*

$$\varepsilon = \frac{1}{Re} \ll 1.$$

- For $\varepsilon = 0$ (inviscid flow) the solution is $\psi \equiv y$, which doesn't satisfy the no-slip boundary condition on the plate:

$$u = \frac{\partial \psi}{\partial y} \equiv 1 \neq 0 \text{ on } y = 0,\ x > 0.$$

- For $0 < \varepsilon \ll 1$, expect this solution to apply at leading order except in a thin boundary layer near the plate in which viscosity (via the $\varepsilon \boldsymbol{\nabla}^4 \psi$ term) reduces the $u$ from its free stream value to zero:



Inviscid fluid ($\epsilon = 0$) Viscous fluid ($0 < \epsilon \ll 1$)

- Since there is no known exact solution, we use boundary layer theory.

### 2.2.4 Boundary layer analysis

- In the outer region away from the plate, we expand

$$\psi \sim \psi_0 + \varepsilon \psi_1 + \cdots \quad \Rightarrow \quad \psi_0 = y \qquad \text{(as expected).}$$

- To determine the thickness of the boundary layer $\delta = \delta(\varepsilon)$ on the plate as $\varepsilon \to 0$, we scale $y = \delta Y$.
- Since

$$u = \frac{\partial \psi}{\partial y} \sim 1$$

as $\varepsilon \to 0$ in the outer region, we also scale $\psi = \delta \Psi$.
- The partial differential equation (41) becomes

$$\frac{\delta}{\delta}\frac{\partial \Psi}{\partial Y}\left(\delta \frac{\partial^3 \Psi}{\partial x^3} + \frac{\delta}{\delta^2}\frac{\partial^3 \Psi}{\partial Y^2 \partial x}\right) - \delta \frac{\partial \Psi}{\partial x}\left(\frac{\delta}{\delta}\frac{\partial^3 \Psi}{\partial x^2 \partial Y} + \frac{\delta}{\delta^3}\frac{\partial^3 \Psi}{\partial Y^3}\right) = \varepsilon \delta \frac{\partial^4 \Psi}{\partial x^4} + \frac{2\varepsilon\delta}{\delta^2}\frac{\partial^4 \Psi}{\partial x^2 \partial Y^2} + \frac{\varepsilon\delta}{\delta^4}\frac{\partial^4 \Psi}{\partial Y^4}$$

- Since the LHS is of $\mathrm{O}(1/\delta)$ , while the RHS is of $\mathrm{O}(\varepsilon/\delta^3)$ , it is necessary that $\varepsilon/\delta^3 = \mathrm{O}(1/\delta)$ for a nontrivial balance involving both inertia and viscous terms.
- Hence, we set (without loss of generality) $\delta = \varepsilon^{1/2}$, so that $y = \varepsilon^{1/2} Y$, $\psi = \varepsilon^{1/2}\Psi$ and

$$\varepsilon \frac{\partial \Psi}{\partial Y}\frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial \Psi}{\partial Y}\frac{\partial^3 \Psi}{\partial Y^2 \partial x} - \varepsilon \frac{\partial \Psi}{\partial x}\frac{\partial^3 \Psi}{\partial x^2 \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^3 \Psi}{\partial Y^3} = \varepsilon^2 \frac{\partial^4 \Psi}{\partial x^4} + 2\varepsilon \frac{\partial^4 \Psi}{\partial x^2 \partial Y^2} + \frac{\partial^4 \Psi}{\partial Y^4}.$$

- Expand $\Psi \sim \Psi_0 + \varepsilon \Psi_1 + \cdots$ to obtain at leading order a version of Prandtl's boundary layer equation:

$$\frac{\partial \Psi_0}{\partial Y}\frac{\partial^3 \Psi_0}{\partial Y^2 \partial x} - \frac{\partial \Psi_0}{\partial x}\frac{\partial^3 \Psi_0}{\partial Y^3} = \frac{\partial^4 \Psi_0}{\partial Y^4}. \tag{44}$$

- The boundary conditions on plate (42) imply that

$$\Psi_0 = \frac{\partial \Psi_0}{\partial Y} = 0 \text{ on } Y = 0,\ x > 0. \tag{45}$$

- We consider the boundary layer on the top of the plate, with $Y > 0$.
- Since the outer solution $\psi \sim \psi_0 = y$ as $\varepsilon \to 0$ and $y = \varepsilon^{1/2} Y$, $\psi = \varepsilon^{1/2}\Psi$, the matching condition is

$$\Psi_0 \sim Y \text{ as } Y \to \infty. \tag{46}$$

**Justification for matching condition**

- To ensure that the inner (boundary layer) and outer expansions match (*i.e.* that they coincide in some intermediate overlap region), introduce *e.g.* the intermediate variable

$$\overline{y} = \frac{y}{\varepsilon^\alpha} = \varepsilon^{(1/2-\alpha)} Y,$$

where $0 < \alpha < 1/2$, so that $y \to 0$ and $Y \to \infty$ as $\varepsilon \to 0$ with $\overline{y} = \mathrm{O}(1)$ fixed.

- In the outer region substitute $y = \varepsilon^\alpha \overline{y}$ and expand as $\varepsilon \to 0$ with $\overline{y} = \mathrm{O}(1)$ fixed:

$$\psi(x,\ y) \sim \psi_0(x,\ \varepsilon^\alpha \overline{y}) \sim \varepsilon^\alpha \overline{y}.$$

- In the inner region substitute $Y = \varepsilon^{(\alpha-1/2)} \overline{y}$ and expand as $\varepsilon \to 0$ with $\overline{y} = \mathrm{O}(1)$ fixed:

$$\psi(x,\ y) = \varepsilon^{1/2} \Psi(x,\ Y) \sim \varepsilon^{1/2} \Psi_0(x,\ \varepsilon^{(\alpha-1/2)} \overline{y})\ .$$

- Hence, the expansions agree in the overlap region in which $\overline{y} = \mathrm{O}(1)$ provided

$$\varepsilon^{1/2} \Psi_0(x,\ \varepsilon^{(\alpha-1/2)} \overline{y}) \sim \varepsilon^\alpha \overline{y} \quad \text{as } \varepsilon \to 0,$$

*i.e.*

$$\Psi_0(x,\ \varepsilon^{(\alpha-1/2)} \overline{y}) \sim \varepsilon^{(\alpha-1/2)} \overline{y} \quad \text{as } \varepsilon \to 0,$$

which can only be the case if

$$\Psi_0(x,\ Y) \sim Y \quad \text{as } Y \to \infty.$$

**Further details**:
In the streamfunction formulation, the problem is

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(y, x)} = \epsilon \nabla^4 \psi \tag{41}$$

or

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \epsilon\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4}\right),$$

with

$$\text{on } y = 0, x > 0: \qquad \psi = \frac{\partial \psi}{\partial y} = 0, \tag{42}$$

and the far-field condition

$$\text{as } x^2 + y^2 \to \infty: \qquad \psi \sim y \quad \left(\text{or } \frac{\partial \psi}{\partial y} \sim 1\right), \tag{43}$$

Outer solution:
In $x = O(1), y = O(1)$ we pose

$$\psi = \psi_0(x, y) + \epsilon \psi_1(x, y) + \dots \qquad \text{as } \epsilon \to 0$$

to obtain at $O(\epsilon^0)$:

$$\frac{\partial(\psi_0, \nabla^2\psi_0)}{\partial(y,x)} = 0$$

with

$$\text{as } x^2 + y^2 \to \infty: \qquad \psi_0 \sim y$$

and

$$\text{on } y = 0, x > 0: \qquad \psi_0 = \frac{\partial \psi_0}{\partial y} = 0, \tag{42'}$$

The solution

$$\psi_0 = y \tag{42.1}$$

satsifies this problem apart from the no-slip (i.e. second) condition in (42'). To satisfy this condition, we now scale near to the plate and consider an inner region or boundary layer.

Inner solution:
We scale as follows: $x = x, y = \delta Y, \psi = \theta\Psi$ where $\delta = \delta(\epsilon) \ll 1$ and $\theta = \theta(\epsilon)$ are two gauges to be determined. In $x = O(1), Y = O(1)$ we obtain

$$\frac{\theta^2}{\delta}\frac{\partial \Psi}{\partial Y}\frac{\partial}{\partial x}\left(\frac{\partial^2\Psi}{\partial x^2} + \underline{\frac{1}{\delta^2}\frac{\partial^2\Psi}{\partial Y^2}}\right) - \frac{\theta^2}{\delta}\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial Y}\left(\frac{\partial^2\Psi}{\partial x^2} + \underline{\frac{1}{\delta^2}\frac{\partial^2\Psi}{\partial Y^2}}\right) = \epsilon\theta\left(\frac{\partial^4\Psi}{\partial x^4} + \frac{2}{\delta^2}\frac{\partial^4\Psi}{\partial x^2\partial Y^2} + \underline{\frac{1}{\delta^4}\frac{\partial^4\Psi}{\partial Y^4}}\right). \tag{42.2}$$

The largest terms are shown underlined on each side and balancing these requires

$$\frac{\theta^2}{\delta^3} = \frac{\epsilon\theta}{\delta^4} \implies \theta\delta = \epsilon. \tag{42.3}$$

The plate condition is

$$\text{on } Y = 0, x > 0: \qquad \Psi = \frac{\partial\Psi}{\partial Y} = 0,$$

and the matching condition with the outer

$$\text{as } Y \to \infty, x > 0: \qquad \theta\Psi \sim \lim_{y\to 0}\psi_{\text{outer}} \quad \text{matching } \psi$$

$$\text{or } \frac{\theta}{\delta}\frac{\partial\Psi}{\partial Y} \sim \lim_{y\to 0}\frac{\partial\psi_{\text{outer}}}{\partial y} \quad \text{matching } \frac{\partial\psi}{\partial y}$$

Since $\psi_{\text{outer}} \sim \psi_0 = y = \delta Y$ then we deduce at leading order that matching requires

$$\theta = \delta \tag{42.4}$$

with

$$\Psi \sim Y \quad \text{or} \quad \frac{\partial\Psi}{\partial Y} \sim 1 \qquad \text{as } Y \to \infty.$$

Consequently our dominant balance from (42.3) and (42.4) is given by

$$\delta = \theta = \epsilon^{1/2}.$$

Posing

$$\Psi = \Psi_0(x, Y) + o(1) \qquad \text{as } \epsilon \to 0$$

gives the leading order problem (i.e. at $O(\epsilon^0)$) in $x = O(1), Y = O(1)$:

$$\frac{\partial\Psi_0}{\partial Y}\frac{\partial^3\Psi_0}{\partial x\partial Y^2} - \frac{\partial\Psi_0}{\partial x}\frac{\partial^3\Psi_0}{\partial Y^3} = \frac{\partial^4\Psi_0}{\partial Y^4}. \tag{42.5}$$

$$\text{on } Y = 0, x > 0: \qquad \Psi_0 = \frac{\partial\Psi_0}{\partial Y} = 0, \tag{42.6}$$

$$\text{as } Y \to \infty, x > 0: \qquad \Psi_0 \sim Y \;\; \text{or} \;\; \frac{\partial \Psi_0}{\partial Y} \sim 1. \tag{42.7}$$

The boundary layer equation (42.5) may be integrated wrt $Y$ once to give

$$\frac{\partial \Psi_0}{\partial Y}\frac{\partial^2 \Psi_0}{\partial x \partial Y} - \frac{\partial \Psi_0}{\partial x}\frac{\partial^2 \Psi_0}{\partial Y^2} = \frac{\partial^3 \Psi_0}{\partial Y^3} + a(x), \tag{42.8}$$

where $a(x)$ is arbitrary function of $x$. The matching condition with the outer (42.7) gives

$$\frac{\partial^2 \Psi_0}{\partial Y^2} \to 0, \; \frac{\partial^3 \Psi_0}{\partial Y^2} \to 0,$$

and thus $a(x) = 0$ so that (42.8) becomes

$$\frac{\partial \Psi_0}{\partial Y}\frac{\partial^2 \Psi_0}{\partial x \partial Y} - \frac{\partial \Psi_0}{\partial x}\frac{\partial^2 \Psi_0}{\partial Y^2} = \frac{\partial^3 \Psi_0}{\partial Y^3}, \tag{42.9}$$

Note: For a more general external flow, we may replace the matching condition (42.7) with

$$\text{as } Y \to \infty, x > 0: \qquad \Psi_0 \sim U_s(x) Y \;\; \text{or} \;\; \frac{\partial \Psi_0}{\partial Y} \sim U_s(x),$$

for which then $a(x) = U_s(x)U_s'(x)$. (42.8) then gives the boundary layer equation corresponding to an inviscid flow whose limiting behaviour on the plate $y = 0^+$ is from Bernoulli's equation (or the x-momentum equation)

$$p_0(x) + \frac{1}{2}U_s(x)^2 = \text{constant} \quad \text{or} \quad \frac{dp_0}{dx} = -U_s \frac{dU_s}{dx}$$

where $p_0$ is the leading-order pressure of the inviscid flow near the plate.

## 2.3 Alternative derivation of the boundary layer equations

### 2.3.1 Dimensionless problem

- Recall that the dimensionless two-dimensional steady Navier-Stokes equations are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{47}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \varepsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{48}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{49}$$

  where $\mathbf{u} = u(x,\ y)\mathbf{i} + v(x,\ y)\mathbf{j}$ is the velocity, $p(x,\ y)$ is the pressure and $\varepsilon = 1/Re \ll 1$.

- The no-flux and no-slip boundary conditions on the plate are given by

$$u = 0, \;\; v = 0 \text{ on } y = 0, \;\; x > 0. \tag{50}$$

- The far-field boundary conditions are given by

$$u \to 1, \;\; v \to 0 \text{ as } x^2 + y^2 \to \infty. \tag{51}$$

### 2.3.2 Boundary layer analysis

- If $\varepsilon = 0$, then $u = 1,\ v = 0$; but this solution of the Euler equations doesn't satisfy the no-slip boundary condition on the plate.
- If $0 < \varepsilon \ll 1$, then $u \sim 1,\ v = \mathrm{o}(1)$ away from the plate on both sides of which there is a thin viscous boundary layer.
- To determine the thickness of the viscous boundary layer $\delta = \delta(\varepsilon)$ on the plate as $\varepsilon \to 0$, we scale $y = \delta Y$.
- By (49),

$$\frac{\partial u}{\partial x} + \frac{1}{\delta}\frac{\partial v}{\partial Y} = 0,$$

  so we scale $v = \delta V$ for a nontrivial balance.
- By (47),

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\varepsilon}{\delta^2}\frac{\partial^2 u}{\partial Y^2},$$

  so we set $\delta = \varepsilon^{1/2}$ for a nontrivial balance involving both inertia and viscous terms.
- Hence, the boundary layer scalings are $y = \varepsilon^{1/2}Y,\ v = \varepsilon^{1/2}V$ and (47)-(49) become

$$\begin{aligned}
u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} &= -\frac{\partial p}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial Y^2},\\
\varepsilon\left(u\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial Y}\right) &= -\frac{\partial p}{\partial Y} + \varepsilon^2\frac{\partial^2 V}{\partial x^2} + \varepsilon\frac{\partial^2 V}{\partial Y^2},\\
\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} &= 0.
\end{aligned}$$

- Expanding

$$u \sim u_0 + \varepsilon u_1 + \cdots, \quad V \sim V_0 + \varepsilon V_1 + \cdots, \quad p \sim p_0 + \varepsilon p_1 + \cdots,$$

  we obtain at leading order Prandtl's boundary layer equations

$$u_0\frac{\partial u_0}{\partial x} + V_0\frac{\partial u_0}{\partial Y} = -\frac{\partial p_0}{\partial x} + \frac{\partial^2 u_0}{\partial Y^2}, \tag{52}$$

$$0 = -\frac{\partial p_0}{\partial Y}, \tag{53}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial V_0}{\partial Y} = 0. \tag{54}$$

- The no-flux and no-slip boundary conditions on the plate (50) are unchanged at leading order, *i.e.*

$$u_0 = 0,\ V_0 = 0 \text{ on } Y = 0,\ x > 0. \tag{55}$$

- The far-field boundary conditions (51) are replaced by the matching condition that

$$u_0 \to 1 \text{ as } |Y| \to \infty,\ x > 0, \tag{56}$$

  which ensures that the leading-order solutions in the outer region (away from the plate) and in the inner boundary-layer region (on the plate) are in agreement in an intermediate 'overlap' region between them (matching procedure not examinable).
- Note that a matching condition is not required for $V_0$ because there are no terms involving the second-order derivatives of $V_0$ in the boundary layer equations (52)–(54) .

### 2.3.3 Implications of the matching condition

- By (53), the pressure does not vary across the boundary layer at leaching order, *i.e.*$p_0 = p_0(x)$ .
- By (56),

$$\frac{\partial u_0}{\partial Y} \to 0, \quad \frac{\partial^2 u_0}{\partial Y^2} \to 0 \qquad \text{as } |Y| \to \infty,$$

so taking the limit $|Y| \to \infty$ in (52) we deduce that

$$\frac{\partial p_0}{\partial x} \to 0 \quad \text{as } |Y| \to \infty;$$

since $p_0$ is independent of $Y$,

$$\frac{\partial p_0}{\partial x} \equiv 0.$$

### 2.3.4 Streamfunction formulation

- By (54), there exists a streamfunction $\Psi_0(x,\ Y)$ such that

$$u_0 = \frac{\partial \Psi_0}{\partial Y}, \quad V_0 = -\frac{\partial \Psi_0}{\partial x},$$

in terms of which (52), with $\partial p_0/\partial x = 0$, becomes

$$\frac{\partial \Psi_0}{\partial Y}\frac{\partial^2 \Psi_0}{\partial x \partial Y} - \frac{\partial \Psi_0}{\partial x}\frac{\partial^2 \Psi_0}{\partial Y^2} = \frac{\partial^3 \Psi_0}{\partial Y^3}. \tag{57}$$

- The boundary conditions (55) become (setting $\Psi_0 = 0$ on the plate, without loss of generality)

$$\Psi_0 = 0, \quad \frac{\partial \Psi_0}{\partial Y} = 0 \quad \text{on } Y = 0 < x. \tag{58}$$

- The matching condition (56) becomes

$$\Psi_0 \sim Y \quad \text{as } |Y| \to \infty. \tag{59}$$

## 2.4 Blasius' similarity solution

- Note that for all $\alpha > 0$ the boundary-layer problem (57)–(59) is invariant under the transformation

$$x \mapsto \alpha^2 x, \quad Y \mapsto \alpha Y, \quad \Psi_0 \mapsto \alpha \Psi_0. \tag{60}$$

- Hence, the solution (assuming it exists and is unique) can involve $x$, $Y$ and $\Psi_0$ in combinations invariant under this transformation only, *e.g.*

$$\left.\begin{array}{c} \dfrac{\Psi_0}{Y} \\ \dfrac{\Psi_0}{x^{1/2}} \end{array}\right\} \text{ a function of } \left\{\begin{array}{c} \dfrac{x}{Y^2} \\ \dfrac{Y}{x^{1/2}}. \end{array}\right.$$

- We choose to seek a solution of the form

$$\Psi_0 = x^{1/2} f(\eta), \quad \eta = \frac{Y}{x^{1/2}} \tag{61}$$

and use the chain rule to show that (57)–(59) reduces to Blasius' equation

$$f''' + \frac{1}{2} f f'' = 0, \tag{62}$$

with boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \tag{63}$$

where prime/denotes differentiation with respect to $\eta$.

**Remarks**

(i) The existence of the invariant transformation (60) implies the ansatz (61) reduces the partial-differential-equation problem (57)–(59) to the ordinary-differential-equation problem (62)–(63). The proof uses group theory and is beyond scope of course (see *e.g.* Ockendon & Ockendon, Appendix B for details), though we shall use such invariant transformations below to facilitate other similarity reductions.

(ii) The ansatz (61) is called a *similarity solution* because knowledge of $\Psi_0$ at $x = x_0 > 0$ is sufficient to determine $\Psi_0$ for all $x > 0$ by a suitable scaling of $x$, *i.e.* the solution looks geometrically the same at different values of $x$.

**Further details**:
Problem (42.6),(42.7) & (42.9) is scale invariant under the scaling group

$$x = \alpha\hat{x}, \quad Y = \alpha^{1/2}\hat{Y}, \quad \Psi_0 = \alpha^{1/2}\hat{\Psi}_0,$$

for any $\alpha \in \mathbb{R}_+$. Thus

$$\Psi_0(x, Y) = \alpha^{1/2}\hat{\Psi}_0(\hat{x}, \hat{Y}) = \alpha^{1/2}\hat{\Psi}_0\left(\frac{x}{\alpha}, \frac{Y}{\alpha^{1/2}}\right) = x^{1/2}\hat{\Psi}_0\left(1, \frac{Y}{x^{1/2}}\right)$$

choosing $\alpha = x$. This motivates the similarity solution

$$\Psi_0(x, Y) = x^{1/2} f(\eta), \qquad \eta = \frac{Y}{x^{1/2}}.$$

Performing the partial derivatives:

$$\frac{\partial\eta}{\partial x} = -\frac{1}{2}Yx^{-3/2} = -\frac{1}{2}\eta x^{-1}, \qquad \frac{\partial\eta}{\partial Y} = x^{-1/2}$$

$$\frac{\partial\Psi_0}{\partial x} = \frac{1}{2}x^{-1/2}f(\eta) + x^{1/2}f'(\eta)\frac{\partial\eta}{\partial x} = \frac{1}{2}x^{-1/2}f(\eta) + x^{1/2}f'(\eta)Y\left(-\frac{1}{2}x^{-3/2}\right) = \frac{1}{2}x^{-1/2}\left(f - \eta f'\right),$$
$$\frac{\partial\Psi_0}{\partial Y} = x^{1/2}f'(\eta)\frac{\partial\eta}{\partial Y} = x^{1/2}f'x^{-1/2} = f',$$
$$\frac{\partial^2\Psi_0}{\partial Y^2} = f''(\eta)\frac{\partial\eta}{\partial Y} = f''x^{-1/2}, \qquad \frac{\partial^3\Psi_0}{\partial Y^3} = x^{-1}f'''(\eta),$$
$$\frac{\partial^2\Psi_0}{\partial x\partial Y} = \frac{1}{2}x^{-1/2}\left(f - \eta f'\right)'\frac{\partial\eta}{\partial Y} = \frac{1}{2}x^{-1}\left(f' - f' - \eta f''\right) = -\frac{1}{2}x^{-1}\eta f'',$$

we obtain the two-point boundary value problem (BVP):

$$f''' + \frac{1}{2}ff'' = 0, \tag{62}$$

with

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{63}$$

There is no exact solution to this nonlinear third-order ode (although it can itegrated twice to a first-order ode). We are left with seeking a numerical solution, which can be reformulated as an initial value problem (IVP) of solving (62) subject to

$$f(0) = f'(0) = 0, \qquad f''(0) = C,$$

where the parameter $C$ is determined so that the far-field condition $f'(\infty) = 1$ is satisfied. The details are below.

Note: We remark, just as in the thermal case, that the above boundary layer analysis breaksdown near the edge of the plate when $x = O(\epsilon)$ (and $y = O(\epsilon)$), the full equations then holding.

### 2.4.1 Numerical solution of (62)–(63)

- Blasius' equation is a third-order, nonlinear, autonomous ordinary differential equation.
- There are three boundary conditions at different boundaries (two at $\eta = 0$ and one at $\eta = \infty$), so (62)–(63) is a two-point boundary value problem.
- Although there is no explicit solution, the boundary value problem may transformed into an initial value problem and solved numerically, as follows.
- Consider the transformation

$$f(\eta) = \gamma F(\xi), \quad \eta = \xi/\gamma,$$

where $\gamma > 0$ is a constant.

- Since

$$f' = \gamma^2 F', \quad f'' = \gamma^3 F'', \quad f''' = \gamma^4 F''',$$

Blasius' equation (62) becomes

$$F''' + \frac{1}{2}FF'' = 0. \tag{64}$$

- Hence, if we solve (64) subject to the initial conditions

$$F(0) = 0, \quad F'(0) = 0, \quad F''(0) = 1, \tag{65}$$

then $f(\eta)$ satisfies Blasius' equation (62) and the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = \gamma^2 F'(\infty), \tag{66}$$

*i.e.* (62)–(63) provided $\gamma = F'(\infty)^{-1/2}$.

- The initial value problem (64)–(65) may be solved numerically in maple by formulating it as a system of first-order differential equations. It is found that $\gamma = C^{1/3}$, where $f''(0) = \gamma^3 = C \approx 0.332$ is a constant used below.

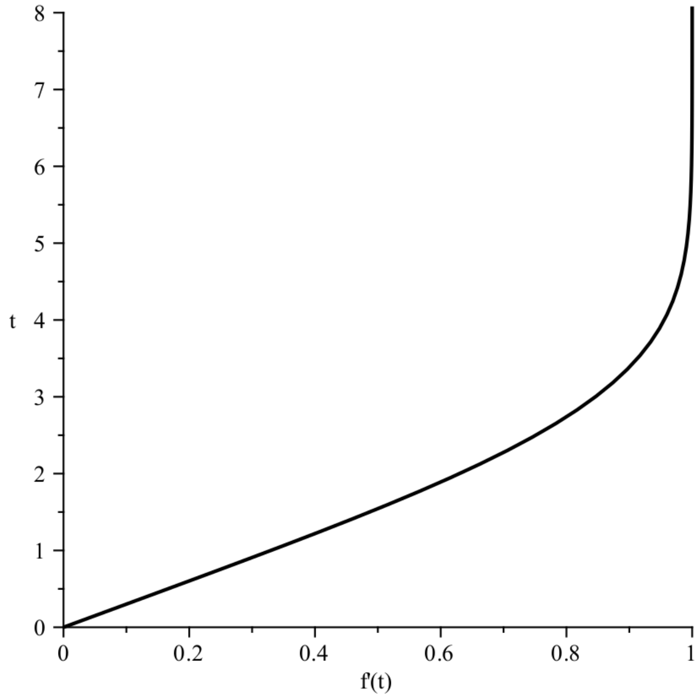
```
> # SOLVE NUMERICALLY BLASIUS INITIAL VALUE PROBLEM

  ODE := diff(F(t),t) = U(t), diff(U(t),t) = V(t), diff(V(t),t) =
  -1/2*F(t)*V(t):
  ICS := F(0)=0, U(0)=0, V(0)=1:
  SOL := dsolve([ODE,ICS],numeric):

  # FIND CONSTANTS

  gamma0 := (rhs(SOL(10)[3]))^(-1/2);
  C := gamma0^3;
  gamma0 := (rhs(SOL(100)[3]))^(-1/2);
  C := gamma0^3;

  # PLOT VELOCITY PROFILE

  with(plots):
  odeplot(SOL,[gamma0^2*U(t),t/gamma0],0..6,color=black,thickness=
  2,view=[0..1,0..8],labels=["f'(t)", "t"]);
```

$\gamma 0 := 0.6924754573$

$C := 0.3320573956$

$\gamma 0 := 0.6924754573$

$C := 0.3320573956$




- This trick was spotted in 1942 by Weyl, who also proved that there exists a unique solution to (62)–(63); 35 years later Blasius' equation (62) was reduced to a first-order ordinary differential equation.

### 2.4.2 Implications

- The dimensional shear stress on the plate is given by

$$\sigma_{12} = \frac{\mu U}{L}\left(\frac{1}{\varepsilon^{1/2}}\frac{\partial u}{\partial Y} + \varepsilon^{1/2}\frac{\partial V}{\partial x}\right)\bigg|_{Y=0}.$$

  Since $V = 0$ on $Y = 0$,

$$\sigma_{12} = \frac{\mu U}{\varepsilon^{1/2}L}\frac{\partial u}{\partial Y}\bigg|_{Y=0} \sim \frac{\mu U}{\varepsilon^{1/2}L}\frac{\partial^2\Psi_0}{\partial Y^2}\bigg|_{Y=0} = \rho\left(\frac{\nu U^3}{Lx}\right)^{1/2} f''(0)\ ,$$

  where $f''(0) = C \approx 0.332$ from above.

- That $\sigma_{12} \to \infty$ as $x \to 0$ reflects the fact that Prandtl's theory is invalid at the leading edge of the plate. To find the solution locally it is necessary to solve the full problem (as in the paradigm heated plate problem).

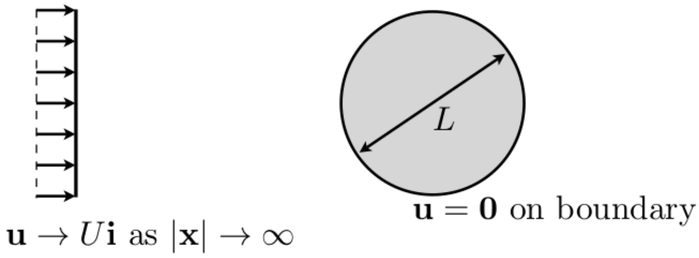
- Ignoring edge singularities, the drag D on one side of a plate of length $L$ is given by

$$D = \int_0^L \sigma_{12}\, d(Lx) = \int_0^1 \sigma_{12} L \mathrm{d}x\ \sim 2f''(0)\rho(\nu U^3 L)^{1/2},$$

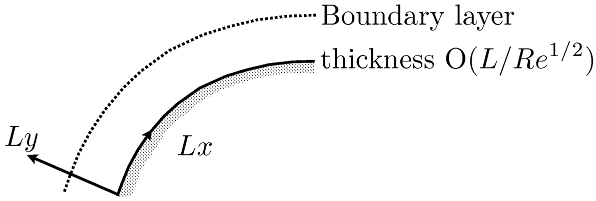
  where $f''(0) = C \approx 0.332$. This prediction compares well with experiment for $10^3 \leq Re \leq 10^5$.

## 2.5 Variable external flow

- Consider the steady two-dimensional flow of a stream of viscous fluid with far-field velocity $U\mathbf{i}$ past an obstacle of typical dimension $L$, with $Re = LU/\nu \gg 1$.




- We might expect a thin boundary layer around the body of thickness of $\mathrm{O}(L/Re^{1/2})$ in which viscous effects are important. This is only partly true, as we shall see.
- On smooth segments of the boundary the boundary layer analysis is the same, but in curvilinear coordinates.
- We denote by $Lx,\ Ly$ the distance along and normal to the surface, so that $x,\ y$ are dimensionless:




- The envisaged boundary-layer scaling is then $y = Y/Re^{1/2}$, where $Y = \mathrm{O}(1)$ as $Re \to \infty$, so that locally $Y = 0$ looks like a flat plate.
- It can be shown that there are no new terms in the boundary layer equations, so the only way the boundary layer "knows" it is on a curved surface is through the matching condition.
- Thus, in the boundary layer the dimensional streamfunction

$$\psi \sim \frac{LU}{Re^{1/2}}\Psi \qquad \text{as } Re \to \infty,$$

where the dimensionless streamfunction $\Psi(x,\ Y)$ satisfies the boundary layer equation

$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial x \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial Y^2} = -\frac{\mathrm{d}p_0}{\mathrm{d}x} + \frac{\partial^3 \Psi}{\partial Y^3}, \tag{67}$$

with boundary conditions

$$\Psi = \frac{\partial \Psi}{\partial Y} = 0 \qquad \text{on } Y = 0, \tag{68}$$

and the matching condition

$$u_0 = \frac{\partial \Psi}{\partial Y} \to U_s(x) \qquad \text{as } Y \to \infty, \tag{69}$$

where $U_s(x) > 0$ is the dimensionless slip velocity predicted by the leading-order-outer inviscid theory.
- As $Y \to \infty$, the matching condition (69) implies that

$$\frac{\partial^2 \Psi}{\partial x \partial Y} \to \frac{\mathrm{d}U_s}{\mathrm{d}x}, \qquad \frac{\partial^2 \Psi}{\partial Y^2} \to 0, \qquad \frac{\partial^3 \Psi}{\partial Y^3} \to 0.$$

- Hence, taking the limit $Y \to \infty$ in (67) gives

$$\frac{\mathrm{d}p_0}{\mathrm{d}x} = -U_s\frac{\mathrm{d}U_s}{\mathrm{d}x}, \tag{70}$$

so that

$$p_0(x) + \frac{1}{2}U_s(x)^2 = \text{constant},$$

which is Bernoulli's equation for the leading-order-outer inviscid flow on $y = 0^+$.

- There is a similarity reduction of the partial differential equation problem (67)–(70) to an ordinary differential equation problem only if
$$U_s(x) \propto (x - x_0)^m \quad \text{or } e^{c(x-x_0)},$$
where $x_0$, $m$ and $c$ are real constants.

### 2.5.1 The Falkner-Skan problem: $U_s(x) = x^m$

- Since the partial-differential-equation problem (67)–(69) with $U_s(x) = x^m$ is invariant to the transformation
$$x \mapsto \alpha x, \quad Y \mapsto \alpha^{(1-m)/2} Y, \quad \Psi \mapsto \alpha^{(1+m)/2}\Psi,$$
for all $\alpha > 0$, we seek a similarity solution of the form
$$\frac{\Psi}{x^{(1+m)/2}} = f(\eta), \qquad \eta = \frac{Y}{x^{(1-m)/2}}.$$
to obtain the Falkner-Skan problem
$$f''' + \frac{1+m}{2} f f'' + m(1 - (f')^2) = 0, \tag{71}$$
with
$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{72}$$

**Remarks**

(i) The Falkner-Skan equation (71) is a third-order nonlinear ordinary differential equation.

(ii) The Falkner-Skan problem (71)–(72) is a two-point boundary value problem that reduces to Blasius' problem for $m = 0$.

(iii) (71) is autonomous, so the order can be lowered by seeking a solution in which $f'$ is the dependent variable and $f$ the independent one.

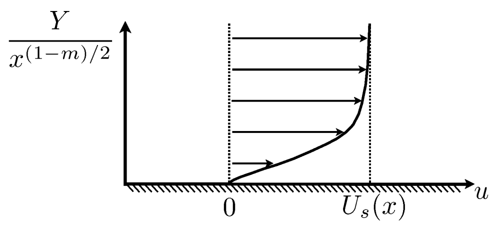
(iv) (71)–(72) can be transformed to an IVP for $m = 0$ only, so the numerics are harder than for Blasius' problem, but amenable to "shooting" methods.

**Numerical Solution of the Falkner-Skan problem (71)–(72)**

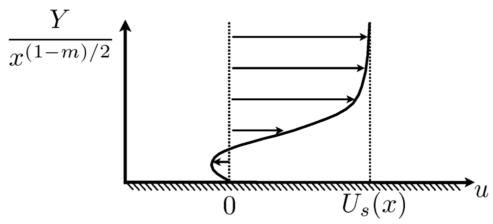
- In the plots below note that in the boundary layer the $x$-component of velocity is given by
$$u_0 = \frac{\partial \Psi}{\partial Y} = x^m f'(\eta), \qquad \eta = \frac{Y}{x^{(1-m)/2}}.$$

- There are three cases, as follows.

  (i) For $m > 0$, there is a unique solution with a monotonic velocity profile:




  (ii) For $-0.0904 < m < 0$, there are two solutions, one with a monotonic velocity profile as in case (i), the other with flow reversal:




(iii) For $m < -0.0904$, there is no solution.

- We return to these results after we have described the physics required to make sense of them.

## 2.6 Breakdown of Prandtl's theory

### 2.6.1 Favourable and adverse pressure gradients

- Assuming $U_s(x) > 0$, so that the slip velocity is in the $x$-direction, the sign of $U_s' = \mathrm{d}U_s/\mathrm{d}x$ determines whether the flow in the boundary layer is driven by

  (i) an accelerating outer flow ($U_s' > 0$) due to a favourable pressure gradient ($p_0' = -U_s U_s' < 0$) ;

  (ii) a decelerating outer flow ($U_s' < 0$) due to an adverse pressure gradient ($p_0' = -U_s U_s' > 0$) .

- **Example**: For $U_s(x) = x^m$, we calculate

$$U_s'(x) = m x^{m-1}, \qquad \frac{\mathrm{d}p_0}{\mathrm{d}x} = -m x^{2m-1},$$

  so the outer flow along the positive $x$-axis is

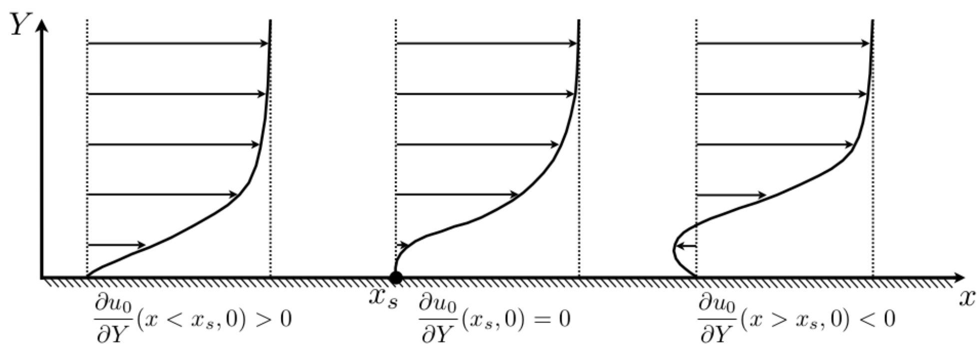
  (i) accelerating and driven by a favourable pressure gradient for $m > 0$;

  (ii) decelerating and driven by an adverse pressure gradient for $m < 0$.

### 2.6.2 Flow reversal

- Numerical simulations of the boundary layer equations show that an adverse pressure gradient ($p_0' > 0$) causes flow reversal near the boundary.

- Flow reversal first occurs at the point of zero skin friction (*i.e.* zero shear stress) on the boundary:




- It may be shown that flow reversal is unstable, so

$$\left.\frac{\partial u_0}{\partial Y}\right|_{Y=0} > 0$$

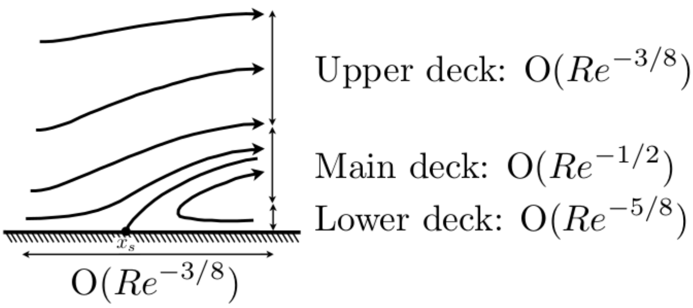
  is a necessary condition for the validity of Prandtl's asymptotic solution (in which viscous effects are confined to a thin-boundary layer on the body).

### 2.6.3 Separation

- If the skin friction vanishes at $x = x_s$, as in the schematic above, then the boundary layer equations break down near $(x_s,\ 0)$ due to the formation of a (Goldstein) singularity, with

$$\frac{\partial^2 u_0}{\partial x \partial Y} \to \infty \quad \text{as } (x,\ Y) \to (x_s,\ 0).$$

- This means that the boundary layer solution cannot be extended into $x > x_s$ (because the boundary layer equations are parabolic, with timelike variable $x$).
- Near $(x_s,\ 0)$ , the asymptotic structure is described by "triple deck theory" (see Acheson §8.6, Ockendon & Ockendon §2.2.4), which predicts separation of the boundary layer from the body at the "separation point" $(x_s,\ 0)$:




- Physical explanation of boundary layer separation (Prandtl 1904): While the free fluid on the edge of the boundary layer has sufficient momentum to traverse an adverse pressure gradient ($U_s > 0,\ U_s' < 0,\ p_0' > 0$) , fluid deep in the boundary layer, having lost a part of its momentum (due to friction), cannot penetrate far into a field of higher pressure, and instead turns away from it.
- Separation sheds vorticity into the outer region, rendering invalid the inviscid irrotational approximation, and hence the whole Prandtl picture.

Potential Flow Recap for following Examples 2.6.4 & 2.6.5:

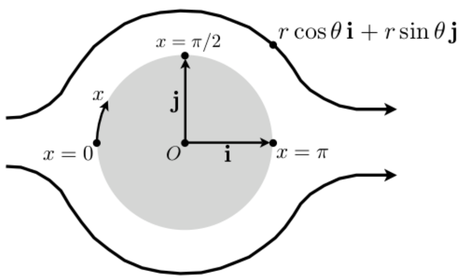
Irrotational flow: $\nabla \wedge \mathbf{u} = 0 \implies \mathbf{u} = \nabla\phi = \dfrac{\partial \phi}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\mathbf{e}_\theta$ for plane polars $(r, \theta)$.

Incompressible flow: $\nabla \cdot \mathbf{u} = 0 \implies \nabla^2 \phi = 0$ or $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = 0.$

Separable solutions: $\phi = f(r)g(\theta) \implies \phi = (Ar^n + Br^{-n})(C\cos(n\theta) + D\sin(n\theta))$.

### 2.6.4 Example: Circular cylinder

- Consider the boundary layer on a circular cylinder of unit radius with far-field velocity $\mathbf{i}$.



- In polar coordinates $(r,\ \theta)$ , the velocity potential

$$\phi(r,\ \theta) = \left(r + \frac{1}{r}\right)\cos\theta$$

if there is no circulation.

Note: The boundary conditions to derive this are:
On $r = 1$: $\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{n} = \mathbf{e}_r \implies \frac{\partial \phi}{\partial r} = 0,$
as $r \to \infty$ : $u = \frac{\partial \phi}{\partial x} \to 1 \implies \phi \to x = r\cos\theta$ (uniform flow).

- Denoting by $x$ the distance along the upper boundary from the forward stagnation point at $-\mathbf{i}$, the slip velocity

$$U_s(x) = \mathbf{u} \cdot (-\mathbf{e}_\theta)\Big|_{\substack{r=1 \\ \theta=\pi-x}} = -\frac{\partial \phi}{\partial \theta}(1,\ \pi - x) = 2\sin x,$$

giving a pressure gradient

$$\frac{\mathrm{d}p_0(x)}{\mathrm{d}x} = -U_s\frac{\mathrm{d}U_s}{\mathrm{d}x} = -2\sin 2x.$$

- Hence, there is

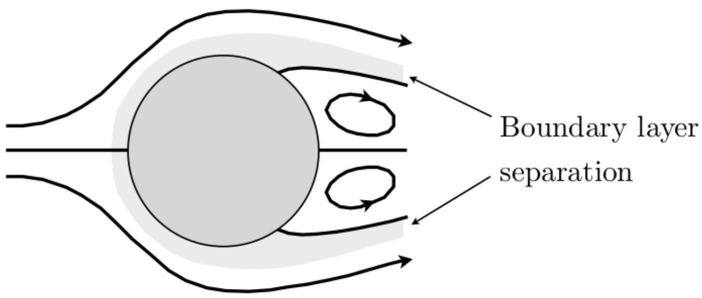
  (i) a favourable pressure gradient between the forward stagnation point at $x = 0$ and a maximum of the slip velocity at $x = \pi/2$;

  (ii) an adverse pressure gradient between $x = \pi/2$ and the rear stagnation point at $x = \pi$.

- A numerical simulation of Prandtl's boundary layer equations shows that flow reversal, and hence separation, occurs at $x = x_s \approx 1.815$, just downstream of the onset of the adverse pressure gradient at $x = \pi/2 \approx 1.571$.

- This completely destroys the Prandtl picture.

- In practice separation occurs for $Re \approx 5 - 10$, with two circulating vortex pairs in the wake:




- As the Reynolds number is increased, the flow in the wake becomes unstable at $Re \approx 10^2$ and turbulent at $Re \approx 10^5$

- The flow up to separation is only slightly affected in the steady laminar regime $(Re \leq 10^2)$ , the prediction of the separation point $x_s \approx 1.815$ being quite good.

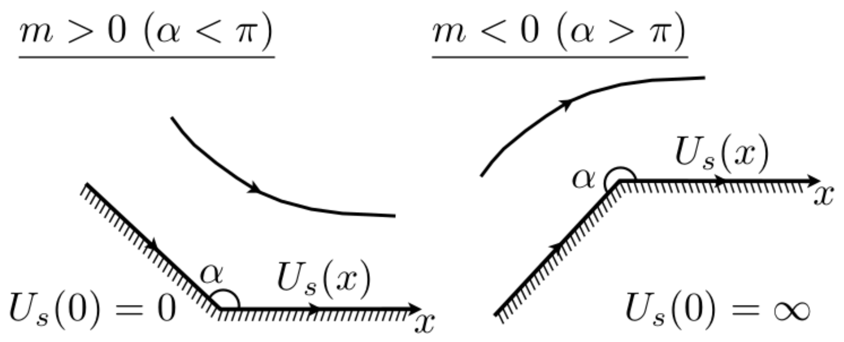
- Deep in the turbulent regime at $Re \geq 3.5 \times 10^6$, the boundary layer itself becomes turbulent and the separation point moves toward the rear stagnation point. The result is a sudden reduction in the size of the wake and hence the drag on the cylinder-this is called the "drag crisis."

### 2.6.5 Example: The theory of flight

- Note that $U_s(x) = x^m$ corresponds to flow past a corner or a wedge in the outer region.
- In polar coordinates $(r,\ \theta)$ , the velocity potential

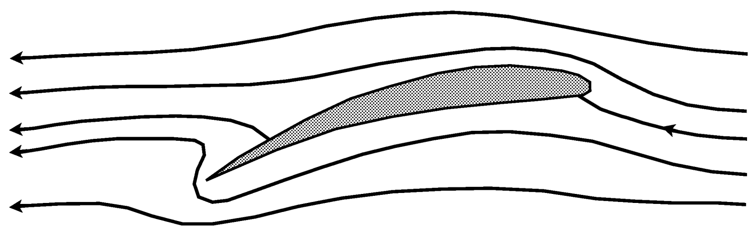
$$\phi(r,\ \theta) = r^{\pi/\alpha} \cos\frac{\pi\theta}{\alpha} \quad \Rightarrow \quad U_s(x) = \mathbf{u}\cdot\mathbf{e}_r\Big|_{y=0} = \frac{\partial\phi}{\partial r}(x,\ 0) \propto x^m,$$

where $m = \pi/\alpha - 1$:

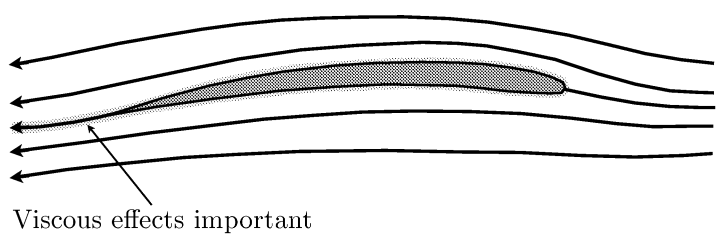



Note: The boundary conditions for $\phi$: On $\theta = 0, \alpha$: $\quad \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{n} = \mathbf{e}_\theta \implies \frac{\partial\phi}{\partial\theta} = 0.$

- This means we can consider the flow in the boundary layer near the sharp trailing edge of an aerofoil.
- If there were no circulation, the outer inviscid irrotational flow would "turn the corner" at the trailing edge:



- On the upper surface of the aerofoil, Prandlt's boundary layer problem near the trailing edge would be the same as the Falkner-Skan problem with $m \approx -1/2$ (as $m = \pi/\alpha - 1$ and $\alpha$ is close to $2\pi$).
- Since $m < 0$, there would be a strong adverse pressure gradient as the flow turns the corner, which would result in separation of the boundary layer from the aerofoil.
- Such a flow is physically unrealistic, an experimental observation consistent with there being no solution to the Falkner-Skan problem for $m < -0.0904$.
- Thus, boundary layer theory provides theoretical justification for the Kutta-Joukowski hypothesis, which says that the appropriate outer inviscid irrotational solution has a circulation just sufficient for the flow to separate smoothly at the trailing edge (*i.e.* with finite velocity).
- In practice it is this solution that describes laminar flow past a blunt nosed aerofoil at a small angle of attack, since the thin viscous wake left behind the aerofoil has a small effect on the outer inviscid flow:

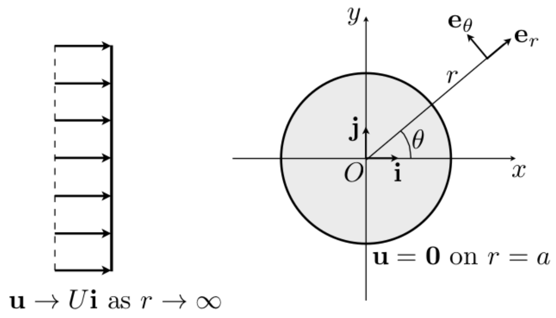


# 3 Low Reynolds number flows

## 3.1 Slow flow past a circular cylinder

### 3.1.1 Dimensional problem

- Consider the two-dimensional steady incompressible viscous flow of a uniform stream $U\mathbf{i}$ past a rigid circular cylinder of radius $a$, centre $O$.
- In the absence of body forces, the flow is governed by the incompressible Navier-Stokes equations (22)–(23) with F $= 0$, which become
$$\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\mathbf{u},\ \ \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$
- The boundary conditions are in the following diagram.




### 3.1.2 Nondimensionalization

- In the slow-flow regime (see §1.11), we nondimensionalize by scaling
$$\mathbf{x} = a\mathbf{x}^*, \quad \mathbf{u} = U\mathbf{u}^*, \quad p = \frac{\mu U}{a}p^*.$$
to obtain (dropping the stars $*$)
$$\varepsilon(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \boldsymbol{\nabla}^2\mathbf{u}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$
where the Reynolds number
$$\varepsilon \ = Re = \frac{\rho U a}{\mu}.$$

### 3.1.3 Streamfunction formulation

- In two dimensions the incompressibility condition implies the existence of a streamfunction $\psi$ independent of $z$ such that
$$\mathbf{u} = \boldsymbol{\nabla}\wedge(\psi\mathbf{k}) \equiv \mathrm{curl}(\psi\mathbf{k}) = u\mathbf{i} + v\mathbf{j} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta.$$
where
$$u = \frac{\partial\psi}{\partial y}, \quad v \ = -\frac{\partial\psi}{\partial x}; \qquad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{\partial\psi}{\partial r}.$$
- Hence the vorticity
$$\begin{aligned}\boldsymbol{\omega} &= \mathrm{curl}(\mathbf{u})\\ &= \mathrm{curl}^2(\psi\mathbf{k})\\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot(\psi\mathbf{k})) - \boldsymbol{\nabla}^2(\psi\mathbf{k})\\ &= \boldsymbol{\nabla}\left(\frac{\partial\psi}{\partial z}\right) - (\boldsymbol{\nabla}^2\psi)\mathbf{k}\\ &= -(\boldsymbol{\nabla}^2\psi)\mathbf{k}.\end{aligned}$$

- We take the curl of the momentum equation to eliminate $p$, and obtain thereby the two-dimensional steady vorticity transport equation

$$\varepsilon(\mathbf{u}\cdot\boldsymbol{\nabla})\omega = \boldsymbol{\nabla}^2\omega$$

for the $z$-component of vorticity $\omega = -\boldsymbol{\nabla}^2\psi$.

- In Cartesian coordinates $(x,\ y)$ , with $\psi = \psi(x,\ y)$ ,

$$\mathbf{u}\cdot\boldsymbol{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}$$

so that

$$\varepsilon\frac{\partial(\psi,\boldsymbol{\nabla}^2\psi)}{\partial(y,x)} = \boldsymbol{\nabla}^2(\boldsymbol{\nabla}^2\psi) \equiv \boldsymbol{\nabla}^4\psi,$$

where

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

- In plane polar coordinates $(r,\ \theta)$ , with $\psi = \psi(r,\ \theta)$ ,

$$\mathbf{u}\cdot\boldsymbol{\nabla} = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta},$$

so that

$$\frac{\varepsilon}{r}\frac{\partial(\psi,\boldsymbol{\nabla}^2\psi)}{\partial(\theta,r)} = \boldsymbol{\nabla}^4\psi,$$

where

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

### 3.1.4 Dimensionless problem

- We have derived the streamfunction formulation

$$\varepsilon\frac{\partial(\psi,\boldsymbol{\nabla}^2\psi)}{\partial(y,x)} = \boldsymbol{\nabla}^4\psi \qquad \text{or} \qquad \frac{\varepsilon}{r}\frac{\partial(\psi,\boldsymbol{\nabla}^2\psi)}{\partial(\theta,r)} = \boldsymbol{\nabla}^4\psi \tag{73}$$

of the dimensionless two-dimensional steady incompressible Navier-Stokes equations, and will consider below the slow-flow regime in which the Reynolds number $\varepsilon$ is small.

- The no-flux and no-slip boundary conditions on the cylinder become

$$\psi = \frac{\partial\psi}{\partial r} = 0 \quad \text{on } r = 1, \tag{74}$$

where we set $\psi = 0$ on $r = 1$ without loss of generality; note that $r = 1$ is a streamline.

- The far-field condition becomes

$$\psi \sim y = r\sin\theta \quad \text{as } r \to \infty. \tag{75}$$

### 3.1.5 Asymptotic solution for $r = \mathrm{O}(1)$ as $\varepsilon \to 0$

- We seek a regular perturbation solution to (73)-(75) by expanding

$$\psi \sim \psi_0 + \varepsilon\psi_1 + \cdots \ \text{as } \varepsilon \to 0.$$

- We obtain at leading-order the slow-flow approximation in which the inertia terms are neglected, *i.e.* the biharmonic equation

$$\boldsymbol{\nabla}^4\psi_0 = 0.$$

- We would like to solve this fourth-order partial differential equation subject to the boundary conditions (76)-(75), which become

$$\psi_0 = \frac{\partial \psi_0}{\partial r} = 0 \quad \text{on } r = 1,$$
$$\psi_0 \sim r \sin\theta \quad \text{as } r \to \infty.$$

- The far-field condition suggests we seek a separable solution of the form

$$\psi_0(r,\ \theta) = f(r) \sin\theta,$$

which implies that

$$\boldsymbol{\nabla}^2 \psi_0 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) f(r)\sin\theta = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \right) f(r)\sin\theta,$$

and hence that

$$\boldsymbol{\nabla}^4 \psi_0 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \right)^2 f(r)\sin\theta.$$

- Thus, $f(r)$ satisfies the fourth-order linear ordinary differential equation

$$\left( \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2} \right)^2 f(r) = 0, \tag{76}$$

with boundary conditions

$$f(1) = f'(1) = 0, \quad f'(\infty) = 1. \tag{77}$$

- The ordinary differential equation (76) is homogeneous because it is invariant under the transformation $r \to \alpha r$ $(\alpha \neq 0)$ , so we seek solutions of the form $f(r) = r^n$ to obtain

$$\left( \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2} \right) f(r) = [n(n-1) + n - 1] r^{n-2} = [n^2 - 1] r^{n-2}$$

so that

$$\left( \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2} \right)^2 f(r) = [n^2 - 1][(n-2)^2 - 1] r^{n-4}.$$

- Hence $n = -1, 1, 1, 3$, so (76) has general solution

$$f(r) = \frac{A}{r} + Br + Cr\log r + Dr^3,$$

where $A,\ B,\ C$ and $D$ are constants.

- Note that this general solution may also be derived by making $s = \log r$ the independent variable.

- By the boundary conditions on the cylinder in (77),

$$f(1) = 0 \quad \Rightarrow \quad A + B + D = 0,$$
$$f'(1) = 0 \quad \Rightarrow \quad -A + B\ + C + 3D = 0,$$

giving

$$C = 4A + 2B, \qquad D = -(A + B).$$

- Applying the far-field condition in (77),

$$f'(\infty) = 1 \quad \Rightarrow \quad B = 1,\ C = 0,\ D = 0.$$

- But

$$C = 0,\ D = 0 \quad \Rightarrow \quad A = 0,\ B = 0,$$

so it is not possible to satisfy both the boundary conditions on the cylinder and the far-field condition.

- Hence, there is no separable solution of the form $\psi(r,\ \theta) = f(r)\sin\theta$.
- Can prove rigorously nonexistence of a solution for slow flow past a cylinder of arbitrary cross-section.
- This is known as the Stokes paradox (1851), which wasn't resolved until 1957!

### 3.1.6 Resolution of Stokes' paradox

- The key observation that facilitates the resolution of the Stokes paradox is that (73)–(75) is a singular perturbation problem because the slow-flow approximation is only valid for

$$r \ll \frac{1}{\varepsilon}.$$

Further details: The full equations are

$$\epsilon\, \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nabla^2\mathbf{u}, \qquad \epsilon = Re,$$

in the steady case. Since $\mathbf{u} = O(1)$ (see the incoming flow condition), the intertia terms are of size $O(\epsilon/r)$ and the viscous terms of $O(1/r^2)$. Thus they are comparable when $\epsilon r = O(1)$ i.e. $r = O(1/\epsilon)$. The slow flow approximation (i.e. neglecting the inertia terms) only holds for $r \ll 1/\epsilon$ and thus breaksdown when $r = O(1/\epsilon)$.

- Hence, instead of imposing the far-field condition

$$\psi_0 \sim r\sin\theta \quad \text{as } r \to \infty,$$

we need to match with a "boundary layer at infinity"

- To balance inertia and viscous terms we scale

$$x = \frac{\hat{x}}{\varepsilon},\ y = \frac{\hat{y}}{\varepsilon},\ r = \frac{\hat{r}}{\varepsilon},\ \psi = \frac{\hat{\psi}}{\varepsilon},$$

so that at first it appears we recover the full two-dimensional formulation:

$$\frac{\partial(\hat{\psi}, \hat{\boldsymbol{\nabla}}^2\hat{\psi})}{\partial(\hat{y}, \hat{x})} = \hat{\boldsymbol{\nabla}}^4\hat{\psi}.$$

Further details: We now are considering the region $r = O(1/\epsilon)$ through the given scalings. The scaling for $\psi$ following from the far-field condition (75) (or equivalently noting that $\psi$ must scale like $x$ or $y$ for $\mathbf{u} = O(1)$ i.e. using $u = \frac{\partial\psi}{\partial y} = O(1), v = -\frac{\partial\psi}{\partial x} = O(1)$).

- However, far away the cylinder has a small effect on the flow, so that

$$\hat{\psi} \sim \hat{y} + \delta\hat{\psi}_1 + \cdots,$$

where $\delta(\varepsilon) \ll 1$ is to be determined.

Further details: The gauge $\delta(\epsilon)$ has to be found and the first term arises due to the far-field condition (75). Introducing the expansion into the equation for $\hat{\psi}$ gives

$$\left(1 + \delta\frac{\partial\hat{\psi}_1}{\partial\hat{y}} + \dots\right)\left(\delta\frac{\partial\hat{\nabla}^2\hat{\psi}_1}{\partial\hat{x}} + O(\delta^2)\right) - O(\delta^2) = \delta\hat{\nabla}^4\hat{\psi}_1.$$

- Obtain at O($\delta$) Oseen's equation

$$\frac{\partial(\hat{y}, \hat{\boldsymbol{\nabla}}^2\hat{\psi}_1)}{\partial(\hat{y},\hat{x})} = \hat{\boldsymbol{\nabla}}^4\hat{\psi}_1$$

or

$$\frac{\partial}{\partial \hat{x}}\hat{\boldsymbol{\nabla}}^2\hat{\psi}_1 = \hat{\boldsymbol{\nabla}}^4\hat{\psi}_1$$

or

$$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{x}} = -\hat{\boldsymbol{\nabla}}\hat{p} + \hat{\boldsymbol{\nabla}}^2\hat{\mathbf{u}}, \qquad \hat{\boldsymbol{\nabla}}\cdot\hat{\mathbf{u}} = 0 \tag{78}$$

upon expanding $\mathbf{u} \sim \mathbf{i} + \delta\hat{\mathbf{u}} + \cdots$, $p \sim \varepsilon\delta\hat{p}$, so that

$$\hat{\mathbf{u}} \ = \frac{\partial\hat{\psi}_1}{\partial\hat{y}}\mathbf{i} - \frac{\partial\hat{\psi}_1}{\partial\hat{x}}\mathbf{j}.$$

- There is no closed form solution to Oseen's equations (78), but can use Fourier transforms to show that the relevant solution (with zero flow at infinity) has the local expansion

$$\hat{\psi}_1 \sim E\hat{r}\log\hat{r}\sin\theta \qquad \text{as } \hat{r}\to 0,$$

where $E$ is an arbitrary constant.

- It is then necessary to match the asymptotic expansions for $\psi$ in the inner region (near the cylinder, with $r = \mathrm{O}(1)$) and in the outer region (far from the cylinder, with $r = \hat{r}/\varepsilon$, $\hat{r} = \mathrm{O}(1)$) by ensuring the constants $A$, $B$, $C$, $D$ and $E$ are such that the expansions are in agreement in an intermediate 'overlap' region between them.

**Matching details**

- Further reading: *Perturbation Methods* by E.J. Hinch; *Perturbation Methods in Fluid Mechanics* by M. Van Dyke.

- To ensure that the inner and outer expansions match ( *i.e.* that they coincide in some overlap region), introduce the intermediate variable

$$\overline{r} = \varepsilon^{\alpha} r = \frac{\hat{r}}{\varepsilon^{1-\alpha}} \quad (0 < \alpha < 1)\ ,$$

so that $r\to\infty$ and $\hat{r}\to 0$ as $\varepsilon\to 0$ with $\overline{r} = \mathrm{O}(1)$ fixed.

- Recall that the leading-order inner solution

$$\psi_0 = \left(\frac{A}{r} + Br + Cr\log r + Dr^3\right)\sin\theta$$

satisfies the boundary conditions on the cylinder $r = 1$ provided

$$C = 4A + 2B, \quad D = -(A+B)\ .$$

- In the inner region fix $\overline{r}$ and expand as $\varepsilon\to 0$:

$$\begin{aligned}\psi &\sim \psi_0(\varepsilon^{-\alpha}\overline{r},\ \theta) + \cdots \\ &\sim D\left(\frac{\overline{r}}{\varepsilon^{\alpha}}\right)^3\sin\theta + C\frac{\overline{r}}{\varepsilon^{\alpha}}\log\left(\frac{\overline{r}}{\varepsilon^{\alpha}}\right)\sin\theta + \cdots \\ &\sim \frac{1}{\varepsilon^{3\alpha}}(D\overline{r}^3\sin\theta) + \frac{1}{\varepsilon^{\alpha}}(-\alpha C\log\varepsilon)\overline{r}\sin\theta + \cdots .\end{aligned}$$

- In the outer region fix $\overline{r}$ and expand as $\varepsilon \to 0$:

$$\begin{aligned}\psi &\sim \frac{1}{\varepsilon}\left(\hat{r}\sin\theta + \delta\hat{\psi}_1 + \cdots\right)\Big|_{\hat{r}=\varepsilon^{1-\alpha}\overline{r}} \\ &\sim \frac{\varepsilon^{1-\alpha}\overline{r}}{\varepsilon}\sin\theta + \frac{\delta}{\varepsilon}E\varepsilon^{1-\alpha}\overline{r}\log(\varepsilon^{1-\alpha}\overline{r})\sin\theta + \cdots \\ &\sim \frac{1}{\varepsilon^{\alpha}}(1 + (1-\alpha)\delta E\log\varepsilon)\overline{r}\sin\theta + \cdots .\end{aligned}$$

- The expansions agree in the overlap region in which $\overline{r} = \mathrm{O}(1)$ provided

$$D = 0, \quad -\alpha C\log\varepsilon = 1 + (1-\alpha)\delta E\log\varepsilon.$$

- This is true for all $0 < \alpha < 1$ provided

$$C = \delta E = \frac{1}{\log(1/\varepsilon)} \ll 1.$$

- Hence, the correct expansion for the streamfunction in the inner region in which $r = \mathrm{O}(1)$ is given by

$$\psi \sim \frac{1}{\log(1/\varepsilon)}\left(r\log r - \frac{r}{2} + \frac{1}{2r}\right)\sin\theta$$

as $\varepsilon \to 0$.

- This asymptotic expansion for $\psi$ proceeds in powers of $1/\log(1/\varepsilon)$ , rather than in powers of $\varepsilon$ as originally anticipated, so very slow convergence.

Alternative Matching details:
Outer solution for $\hat{r} = O(1)$ or $r = O(1/\epsilon)$:

$$\begin{aligned}\psi = \psi_{outer} = \frac{\hat{\psi}}{\epsilon} &= \frac{\hat{y}}{\epsilon} + \frac{\delta}{\epsilon}\hat{\psi}_1 + \dots \\ &\sim \frac{\hat{y}}{\epsilon} + \frac{\delta}{\epsilon}E\hat{r}\ln\hat{r}\sin\theta + \dots \qquad \text{as } \hat{r}\to 0 \\ &= y + \delta Er\ln(\epsilon r)\sin\theta + \dots \qquad \text{(in inner variables, } \hat{r} = \epsilon r) \\ &= r\sin\theta + \delta Er\ln(\epsilon r)\sin\theta + \dots \qquad \text{(all in polars)} \\ &= (\delta Er\ln r + (1 + \delta E\ln\epsilon)r)\sin\theta + \dots \qquad \text{(ordered for } r \text{ large)} \qquad \text{(Mout)}\end{aligned}$$

Inner solution for $r = O(1)$:

$$\begin{aligned}\psi = \psi_{inner} &= \psi_0 + \dots \\ &= \left(Ar^{-1} + Br + Cr\ln r + Dr^3\right)\sin\theta + \dots \\ &\sim \left(Dr^3 + Cr\ln r + Br + Ar^{-1}\right)\sin\theta + \dots \qquad \text{as } r\to\infty \text{ (ordered for } r \text{ large)} \qquad \text{(Min)}\end{aligned}$$

For matching

$$\lim_{r\to\infty}\psi_{inner} = \lim_{\hat{r}\to 0}\psi_{outer}.$$

Comparing (Mout) with (Min) gives

$$D = 0, \qquad C = \delta E, \qquad B = 1 + \delta E\ln\epsilon.$$

However, from the boundary conditions on the cylinder for the inner solution we have

$$C = 4A + 2B, \qquad D = -(A + B).$$

Thus

$$B = -\frac{1}{2}C, \qquad A = \frac{1}{2}C$$

and further

$$-\frac{1}{2}C = 1 + C\ln\epsilon \implies C \sim -\frac{1}{\ln\epsilon}.$$

Since $C = \delta E$, we take

$$\delta = -\frac{1}{\ln\epsilon}, \qquad E = 1.$$

Thus the leading order streamfunction for the inner region $r = O(1)$ is

$$\psi \sim \delta\left(r\ln r - \frac{r}{2} + \frac{1}{2r}\right)\sin\theta \qquad \text{with } \delta = -\frac{1}{\ln\epsilon}\ .$$

### 3.1.7 Exercise: Drag calculation

(i) Show that the dimensionless slow-flow approximation may be written in the form $\boldsymbol{\nabla} p = -\operatorname{curl}(\omega\mathbf{k})$, where $\omega = -\boldsymbol{\nabla}^2\psi$, so that in plane polar coordinates

$$\frac{\partial p}{\partial r} = -\frac{1}{r}\frac{\partial\omega}{\partial\theta}, \qquad \frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{\partial\omega}{\partial r}.$$

Further details: Use the streamfunction form $\psi = \delta f(r)\sin\theta$ to calculate the vorticity as

$$\omega = -\delta\frac{2}{r}\sin\theta$$

and the velocity $\mathbf{u} = \nabla\wedge(\psi\mathbf{k})$ with components

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \delta\frac{f(r)}{r}\cos\theta, \qquad u_\theta = -\frac{\partial\psi}{\partial r} = -\delta f'(r)\sin\theta.$$

The momentum equation may be integrated to give the pressure as

$$p = -\delta\frac{2}{r}\cos\theta + p_\infty,$$

where $p_\infty$ is the constant far-field pressure.

(ii) Hence determine the leading-order terms in the expansions of the stress components

$$\sigma_{rr} = -p + 2\frac{\partial u_r}{\partial r}, \quad \sigma_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta} \quad \text{as } \varepsilon\to 0,\ \text{with } r = \mathrm{O}(1).$$

Further details: Show that

$$\sigma_{rr} = 2\delta\left(\frac{1}{r} + \frac{f'}{r} - \frac{f}{r^2}\right)\cos\theta, \qquad \sigma_{r\theta} = -\delta\left(f'' - \frac{f'}{r} + \frac{f}{r^2}\right)\sin\theta$$

and on the cylinder are

$$\sigma_{rr}|_{r=1} = 2\delta\cos\theta, \qquad \sigma_{r\theta}|_{r=1} = -2\delta\sin\theta, \tag{78.5}$$

using $f(1) = f'(1) = 0$ and $f''(1) = 2$.

(iii) Deduce that the dimensionless drag per unit length on the circular cylinder $r = 1$ is given by

$$\int_0^{2\pi}\sigma_{rr}(1,\ \theta)\cos\theta - \sigma_{r\theta}(1,\ \theta)\sin\theta\,\mathrm{d}\theta \sim \frac{4\pi}{\ln(1/\varepsilon)}.$$

The traction vector on the cylinder is

$$\begin{aligned}
\mathbf{t}(\mathbf{e}_r) &= t_i\mathbf{e}_i = \sigma_{ij}n_j\mathbf{e}_i \underbrace{=}_{\substack{\mathbf{n}=\mathbf{e}_r\\ n_r=1\\ n_\theta=0}} \sigma_{ir}n_r\mathbf{e}_i = \sigma_{rr}\mathbf{e}_r + \sigma_{r\theta}\mathbf{e}_\theta \qquad \text{at } r=1\\
&= \sigma_{rr}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \sigma_{r\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \qquad \text{at } r=1\\
&= (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\mathbf{i} + (\sigma_{rr}\sin\theta + \sigma_{r\theta}\cos\theta)\mathbf{j} \quad \text{at } r=1\\
&= 2\delta\mathbf{i}
\end{aligned}$$

Hence the drag on the cylinder is

$$\mathbf{D} = \int_{r=1} \mathbf{t}(\mathbf{e}_r)\,ds = \int_{\theta=0}^{2\pi} \mathbf{t}(\mathbf{e}_r)rd\theta\Big|_{r=1} = \int_{\theta=0}^{2\pi} 2\delta\mathbf{i}d\theta = 4\pi\delta\mathbf{i}.$$

(iv) Deduce that the dimensional drag per unit length on a circular cylinder is approximately

$$\frac{4\pi\mu U}{\ln\left(\frac{\nu}{Ua}\right)},$$

which depends logarithmically on the radius $a$.

Further details: The dimensional drag is given by

$$D = \underbrace{\frac{\mu U}{a}}_{\substack{\text{stress}\\ \text{scaling}}} \underbrace{\frac{4\pi}{\ln\left(\frac{\mu}{\rho Ua}\right)}}_{\substack{\text{dimenionless}\\ \text{drag}}} \underbrace{a}_{\substack{\text{ds}\\ \text{scaling}}} = \frac{4\pi\mu U}{\ln\left(\frac{\nu}{Ua}\right)}.$$

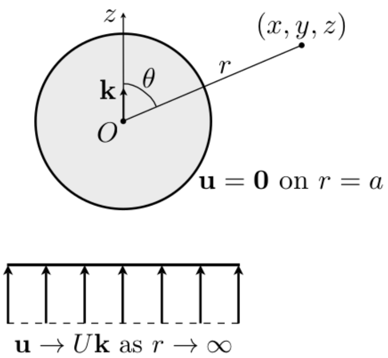
## 3.2 Slow flow past a sphere

### 3.2.1 Dimensional problem

- Consider the three-dimensional steady incompressible viscous flow of a uniform stream $U\mathbf{k}$ past a rigid sphere of radius $a$, centre $O$.
- In the absence of body forces, the flow is governed by the incompressible Navier-Stokes equations (22)-(23) with F $= 0$, which become

$$\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\mathbf{u}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

- The boundary conditions are in the following diagram.



### 3.2.2 Nondimensionalization and slow-flow approximation

- In the slow-flow regime (see §2.9), we nondimensionalize by scaling

$$\mathbf{x} = a\mathbf{x}^*, \quad \mathbf{u} = U\mathbf{u}^*, \quad p = \frac{\mu U}{a}p^*.$$

  to obtain (dropping the stars $*$)

$$Re(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0.$$

  with $\mathbf{u} = 0$ on $r = 1$ and $\mathbf{u} \to \mathbf{k}$ as $r \to \infty$, where the Reynolds number $Re = \rho U a/\mu$.

- For $Re \ll 1$, neglect the inertia term to obtain the slow-flow approximation:

$$\nabla p = \nabla^2\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0.$$

- We begin by rewriting the momentum equation as

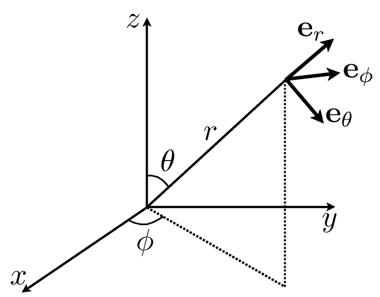
$$\nabla p = \nabla(\nabla\cdot\mathbf{u}) - \nabla\wedge(\nabla\wedge\mathbf{u}) = -\operatorname{curl}^2\mathbf{u}.$$

- Taking the curl we obtain the formulation

$$\operatorname{curl}^3\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{u} = 0.$$

### 3.2.3 Streamfunction formulation

- Introduce spherical polar coordinate system $(r,\ \theta,\ \phi)$ , with origin at the centre of the sphere (Acheson, A.7):




- Seeking a steady axisymmetric solution, with velocity $\mathbf{u} = u_r(r,\ \theta)\mathbf{e}_r + u_\theta(r,\ \theta)\mathbf{e}_\theta$, the incompressibility condition becomes

$$0 = \nabla\cdot\mathbf{u} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(u_r r^2\sin\theta) + \frac{\partial}{\partial\theta}(u_\theta r\sin\theta)\right].$$

- Satisfy this equation exactly by introducing the Stokes streamfunction $\psi(r,\ \theta)$ such that

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

  giving

$$\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & \psi \end{vmatrix} = \operatorname{curl}(\frac{\psi}{r\sin\theta}\mathbf{e}_\phi)$$

- Hence the slow-flow equations reduce to

$$0 = \operatorname{curl}^3\mathbf{u} = \operatorname{curl}^4\left(\frac{\psi}{r\sin\theta}\mathbf{e}_\phi\right)$$

- Calculate:

$$\begin{aligned}
\operatorname{curl}(\mathbf{u}) &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ u_r & ru_\theta & 0 \end{vmatrix} \\
&= \frac{r\sin\theta\mathbf{e}_\phi}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right) \\
&= \frac{\mathbf{e}_\phi}{r}\left(\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \\
&= -\frac{\mathbf{e}_\phi}{r\sin\theta}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right).
\end{aligned}$$

- Hence

$$\operatorname{curl}^2\left(\frac{\psi}{r\sin\theta}\mathbf{e}_\phi\right) = \frac{-D^2\psi}{r\sin\theta}\mathbf{e}_\phi$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \neq \boldsymbol{\nabla}^2.$$

### 3.2.4 Dimensionless problem

- Hence

$$\operatorname{curl}^4\left(\frac{\psi}{r\sin\theta}\mathbf{e}_\phi\right) = \operatorname{curl}^2\left(\frac{-D^2\psi}{r\sin\theta}\mathbf{e}_\phi\right) = \frac{-D^2(-D^2\psi)}{r\sin\theta}\mathbf{e}_\phi,$$

so we will need to solve

$$D^4\psi = 0. \tag{79}$$

- The no-flux and no-slip boundary conditions on the sphere become

$$\psi = \frac{\partial\psi}{\partial r} = 0 \quad \text{on } r = 1, \tag{80}$$

where we set $\psi = 0$ on $r = 1$ without loss of generality.

- As $r \to \infty$,

$$\mathbf{u} \to \mathbf{k} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta,$$

giving

$$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = u_r \to \cos\theta, \qquad -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = u_\theta \to -\sin\theta,$$

so the far-field condition becomes

$$\psi \sim \frac{1}{2}r^2\sin^2\theta \quad \text{as } r \to \infty. \tag{81}$$

### 3.2.5 Stokes solution

- The far-field condition suggests we seek a separable solution of the form

$$\psi = f(r)\sin^2\theta \quad \Rightarrow \quad D^2\psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)f(r)\sin^2\theta \quad \Rightarrow \quad D^4\psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)^2 f(r)\sin^2\theta.$$

- Seeking a solution of the form $f(r) = r^n$ (as for a cylinder) leads to Stokes solution (1851)

$$\psi = \left(\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4}r^{-1}\right)\sin^2\theta,$$

which satisfies the boundary conditions on the sphere and the far-field condition.

- Hence, there is a (separable) solution for slow flow past a sphere, *cf.* slow flow past a cylinder for which the nonexistence of a solution (Stokes paradox) was resolved by matching with a "boundary layer at infinity".
- For slow flow past a sphere nonuniformities arise in the asymptotic solution at $O(Re)$ as $Re \to 0$: this is the Whitehead paradox and it is also resolved by matching with the relevant solution of Oseen's equation in a boundary layer at infinity.
- That the boundary layer at infinity is much weaker for a sphere than for a cylinder is to be expected, as a cylinder is infinitely long.
- At leading order the streamfunction and hence the streamlines are symmetric fore and aft of the sphere (and of the cylinder in the slow flow region), though a wake is observed in practice for moderate Reynolds number.

#### 3.2.6 Drag calculation

- The velocity components are given by

$$u_r = \left(1 - \frac{3}{2}r^{-1} + \frac{1}{2}r^{-3}\right)\cos\theta, \qquad u_\theta = \left(-1 + \frac{3}{4}r^{-1} + \frac{1}{4}r^{-3}\right)\sin\theta.$$

- Using $\boldsymbol{\nabla} p = -\operatorname{curl}^2 \mathbf{u}$, we deduce that the pressure

$$p = p_\infty - \frac{3\cos\theta}{2r^2},$$

where $p_\infty$ is the constant pressure at $\infty$.

- The dimensionless force per unit area on the sphere is given by the dimensionless stress vector (made dimensionless by scaling it with the pressure scale $\mu U/a$)

$$\mathbf{t}(\mathbf{e}_r)|_{r=1} = \sigma_{rr}\mathbf{e}_r + \sigma_{r\theta}\mathbf{e}_\theta|_{r=1}.$$

Note: $\mathbf{t}(\mathbf{e}_r) = t_i\mathbf{e}_i = \sigma_{ij}n_j\mathbf{e}_i \underbrace{=}_{\substack{\mathbf{n}=\mathbf{e}_r, n_r=1 \\ n_\theta=0, n_\phi=0}} \sigma_{ir}n_r\mathbf{e}_i = \sigma_{rr}\mathbf{e}_r + \sigma_{r\theta}\mathbf{e}_\theta + \underbrace{\sigma_{r\phi}}_{=0}\mathbf{e}_\phi$

$\mathbf{e}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}, \quad \mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}.$

- On the sphere the (nonzero) components of the dimensionless stress tensor (made dimensionless by scaling them with the pressure scale $\mu U/a$) are given by

$$\sigma_{rr}|_{r=1} = \left[-p + 2\frac{\partial u_r}{\partial r}\right]_{r=1} = -p_\infty + \frac{3}{2}\cos\theta, \quad \sigma_{r\theta}|_{r=1} = \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]_{r=1} = -\frac{3}{2}\sin\theta.$$

- Hence, the dimensionless force on the sphere is given by

$$\begin{aligned}\iint_{r=1} \mathbf{t}(\mathbf{e}_r)\,\mathrm{d}S &= \int_0^{2\pi}\int_0^{\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)r^2\sin\theta\big|_{r=1}\,\mathrm{d}\theta\mathrm{d}\phi\mathbf{k} \\ &= (2\pi + 4\pi)\mathbf{k} \qquad \text{(normal stress + shear stress)} \\ &= 6\pi\mathbf{k}.\end{aligned}$$

- Recover famous Stokes formula for the dimensional drag:

$$\underbrace{\frac{\mu U}{a}}_{\substack{\text{stress}\\\text{scaling}}}\ \underbrace{6\pi}_{\substack{\text{dimenionless}\\\text{drag}}}\ \underbrace{a^2}_{\mathrm{d}S} = 6\pi\mu U a.$$

- **Example:** Provided $Re = \rho a U/\mu \ll 1$, the terminal velocity $U$ of a solid sphere (uniform density $\rho_s$) falling under gravity $g$ though fluid (density $\rho$, viscosity $\mu$) is given by

$$6\pi\mu U a = \frac{4}{3}\pi(\rho_s - \rho)a^3 g \quad \Rightarrow \quad U = \frac{2}{9}\frac{a^2(\rho_s - \rho)g}{\mu}.$$

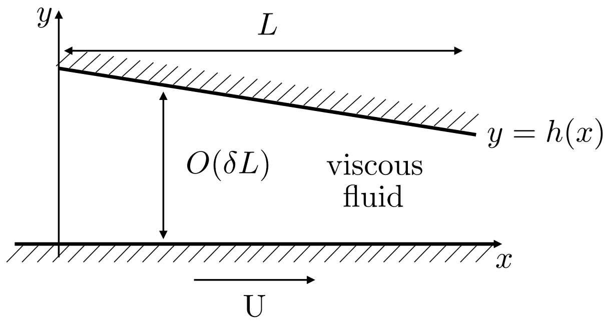
## 3.3 Lubrication Theory

### 3.3.1 Motivation

- How can a thin fluid layer support a large normal load?
- Example: sheet of paper sliding across a table.
- Lubrication theory is the analysis of flows in thin layers whose aspect ratio (depth / length) is small.
- Lubrication theory combines ideas from slow flow and boundary layer theory to simplify the Navier-Stokes equations.
- Applies to relatively low Reynolds number flow.

### 3.3.2 The Slider Bearing

- Consider a 2-d bearing in which the place $y = 0$ moves with constant velocity U in the x-direction. The top of the bearing (the slider) is fixed with viscous fluid between the gap.




- For velocity field $\mathbf{u} = (u, v)$ the steady dimensional Navier-Stokes equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{u}.$$

- Boundary conditions (no-slip and solid boundary/no flux)

$$\text{on } y = 0: \qquad u = U, v = 0,$$

$$\text{on } y = h(x): \qquad u = 0, v = 0.$$

- Let L be the length of the bearing and $H = \delta L$ be the typical gap-width where $\delta \ll 1$ (a small dimensionless parameter) is the aspect ratio $H/L$.

### 3.3.3 Dimensionless problem

- Nondimensionalise:

$$x = L\bar{x}, \quad y = \delta L\bar{y}, \quad h = \delta L\bar{h}, \quad u = U\bar{u}, \quad v = \delta U\bar{v}, \quad p = \underbrace{\rho U^2}_{\substack{\text{inertial}\\ \text{scaling}}} \hat{p},$$

- Dimensionless governing equations:

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} &= 0 \\
\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} &= -\frac{\partial \hat{p}}{\partial \bar{x}} + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{1}{\delta^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right), \\
\delta\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) &= -\frac{1}{\delta}\frac{\partial \hat{p}}{\partial \bar{y}} + \frac{1}{Re}\left(\delta\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{1}{\delta}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right),
\end{aligned}
$$

where the Reynolds number $Re = \rho UL/\mu$.

- Dimensionless boundary conditions:

$$
\begin{aligned}
&\text{on } \bar{y} = 0: && \bar{u} = 1, \bar{v} = 0, \\
&\text{on } \bar{y} = \bar{h}(\bar{x}): && \bar{u} = 0, \bar{v} = 0.
\end{aligned}
$$

- The appropriate scaling of the pressure is clearly seen to be that given by the viscous (force) terms (for a non-trivial balance).

$$
\hat{p} = \frac{1}{\delta^2 Re}\bar{p}, \qquad p = \frac{1}{\delta^2 Re}\,\rho U^2\,\bar{p} = \underbrace{\frac{\mu U}{\delta^2 L}}_{\substack{\text{viscous}\\ \text{scaling}}}\,\bar{p}.
$$

so that the dimensionless governing equations can be written

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} &= 0 \\
Re\delta^2\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \delta^2\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}, \\
Re\delta^4\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{y}} + \delta^2\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \delta^4\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}.
\end{aligned}
$$

#### 3.3.4 Lubrication approximation

- Assuming the reduced Reynolds number is small $Re\delta^2 \ll 1$, as well as a small aspect ratio $\delta \ll 1$, gives at leading order the lubrication model

$$
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{82}
$$

$$
0 = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{83}
$$

$$
0 = -\frac{\partial \bar{p}}{\partial \bar{y}}, \tag{84}
$$

with boundary conditions

$$
\text{on } \bar{y} = 0: \qquad \bar{u} = 1, \bar{v} = 0, \tag{85}
$$

$$
\text{on } \bar{y} = \bar{h}(\bar{x}): \qquad \bar{u} = 0, \bar{v} = 0. \tag{86}
$$

### 3.3.5 Reynolds lubrication equation

- Integrating (84) wrt $\bar{y}$ gives

$$\bar{p} = \bar{p}(\bar{x}).$$

- Integrating (83) using (85) and (86) gives

$$\bar{u} = \frac{1}{2}\frac{d\bar{p}}{d\bar{x}}\bar{y}(\bar{y} - \bar{h}) + 1 - \frac{\bar{y}}{\bar{h}}.$$

- Integrating (82) using (85) gives

$$\bar{v} = -\frac{1}{6}\frac{d^2\bar{p}}{d\bar{x}^2}\bar{y}^3 + \frac{1}{4}\bar{y}^2\frac{d}{d\bar{x}}\left(\bar{h}\frac{d\bar{p}}{d\bar{x}}\right) - \frac{1}{2}\frac{\bar{y}^2}{\bar{h}^2}\frac{d\bar{h}}{d\bar{x}}.$$

- Finally using the condition for $\bar{v}$ in (86) gives Reynolds lubrication equation

$$\frac{1}{6}\frac{d}{d\bar{x}}\left(\bar{h}^3\frac{d\bar{p}}{d\bar{x}}\right) = \frac{d\bar{h}}{d\bar{x}}.$$

- This may be derived directly without finding $\bar{v}$ by integrating (82) across the layer

$$\int_0^{\bar{h}(\bar{x})}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}}\right)d\bar{y} = 0 \implies \int_0^{\bar{h}(\bar{x})}\frac{\partial\bar{u}}{\partial\bar{x}}\,d\bar{y} + \underbrace{\bar{v}(\bar{x},\bar{h}(\bar{x}))}_{=0\text{ by }(86)} - \underbrace{\bar{v}(\bar{x},0)}_{=0\text{ by }(85)} = 0 \implies \int_0^{\bar{h}(\bar{x})}\frac{\partial\bar{u}}{\partial\bar{x}}\,d\bar{y} = 0$$

Hence

$$\frac{d}{d\bar{x}}\int_0^{\bar{h}(\bar{x})}\bar{u}(\bar{x},\bar{y})\,d\bar{y} \underbrace{=}_{\substack{\text{Leibniz's}\\\text{rule}}} \int_0^{\bar{h}(\bar{x})}\frac{\partial\bar{u}}{\partial\bar{x}}\,d\bar{y} + \bar{u}(\bar{x},\bar{h}(\bar{x}))\frac{d\bar{h}}{d\bar{x}} = 0$$

Using the expression for $\bar{u}$ and evaluating the integral then gives Reynolds equation.

Note: $\displaystyle\int_0^{\bar{h}(\bar{x})}\bar{u}(\bar{x},\bar{y})\,d\bar{y} = -\frac{1}{12}\bar{h}^3\frac{d\bar{p}}{d\bar{x}} + \frac{\bar{h}}{2}.$

- If $\bar{h}(\bar{x})$ is given, Reynolds lubrication equation is a second order ODE for $\bar{p}(\bar{x})$ requiring two boundary conditions e.g. prescription of $\bar{p}$ at the ends of the bearing (say at $\bar{x} = 0$ and $\bar{x} = 1$, if it is of dimensional length L).

### 3.3.6 Stress tensor

- Dimensional stress tensor:

$$(\sigma_{ij}) = \begin{pmatrix} -p + 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \bullet & -p + 2\mu\dfrac{\partial v}{\partial y}\end{pmatrix} = \frac{\mu U}{L}\begin{pmatrix} -\dfrac{\bar{p}}{\delta^2} + 2\dfrac{\partial\bar{u}}{\partial\bar{x}} & \dfrac{1}{\delta}\dfrac{\partial\bar{u}}{\partial\bar{y}} + \delta\dfrac{\partial\bar{v}}{\partial\bar{x}} \\ \bullet & -\dfrac{\bar{p}}{\delta^2} + 2\dfrac{\partial\bar{v}}{\partial\bar{y}}\end{pmatrix}$$

- The stress exerted by the fluid on the upper surface $\bar{y} = \bar{h}(\bar{x})$ is given by the traction vector $\mathbf{t} = (t_1, t_2)$, where

$$t_i = \sigma_{ij}n_j, \qquad \mathbf{n} = (n_1, n_2) = \left(1 + \delta^2\left(\frac{d\bar{h}}{d\bar{x}}\right)^2\right)^{-\frac{1}{2}}\left(\delta\frac{d\bar{h}}{d\bar{x}}, -1\right).$$

so that

$$t_1 = \sigma_{11}n_1 + \sigma_{12}n_2$$

$$= \frac{\mu U}{L}\left[\left(-\frac{\bar{p}}{\delta^2}+2\frac{\partial \bar{u}}{\partial \bar{x}}\right)\delta\frac{d\bar{h}}{d\bar{x}}-\frac{1}{\delta}\frac{\partial \bar{u}}{\partial \bar{y}}-\delta\frac{\partial \bar{v}}{\partial \bar{x}}\right]\left(1+\delta^2\left(\frac{d\bar{h}}{d\bar{x}}\right)^2\right)^{-\frac{1}{2}}$$

$$= \frac{\mu U}{L}\left[\frac{1}{\delta}\left(-\bar{p}\frac{d\bar{h}}{d\bar{x}}-\frac{\partial \bar{u}}{\partial \bar{y}}\right)+O(\delta)\right],$$

and

$$t_2 = \sigma_{21}n_1 + \sigma_{22}n_2$$

$$= \frac{\mu U}{L}\left[\left(\frac{\partial \bar{u}}{\partial \bar{y}}+\delta^2\frac{\partial \bar{v}}{\partial \bar{x}}\right)\frac{d\bar{h}}{d\bar{x}}+\frac{\bar{p}}{\delta^2}-2\frac{\partial \bar{v}}{\partial \bar{y}}\right]\left(1+\delta^2\left(\frac{d\bar{h}}{d\bar{x}}\right)^2\right)^{-\frac{1}{2}}$$

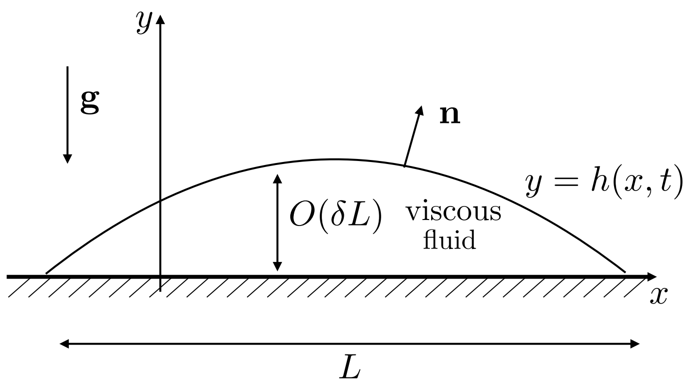
$$= \frac{\mu U}{L}\left[\frac{\bar{p}}{\delta^2}+O(1)\right].$$

- The vertical stress component $t_2$ exerted on the upper surface is an order of magnitude greater than the horizontal stress component $t_1$. This explains why a thin viscous fluid can support large normal loads, whilst offering relatively little shearing (tangential) resistance.

## 3.4 Thin Films with free surfaces

### 3.4.1 Motivation

- Consider gravity-driven flow of a thin two-dimensional viscous layer of *a priori* unknown thickness $h(x,\ t)$ on a stationary horizontal plate at $y = 0$:




- Such a thin film may model spreading of a liquid, molten lava or an ice sheet.
- This is a free boundary problem, as fluid domain $0 < y < h(x,\ t)$ must be determined as part of solution.
- **New features:** need to incorporate gravity into the incompressible Navier-Stokes equations and prescribe appropriate free boundary conditions at the free surface $z = h(x,\ t)$ .
- **Key idea:** assume aspect ratio (depth / length) is small and apply lubrication theory.

### 3.4.2 Dimensional problem

- The 2-d incompressible Navier-Stokes equations for the fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0,$$

$$
\begin{aligned}
\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\
\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) &= -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\rho g.
\end{aligned}
$$

- On the lower fixed surface impose the solid boundary (no-flux) and no-slip boundary conditions

$$
\text{On } y=0 \qquad u=v=0.
$$

- On the free surface we impose the kinematic condition (no-flux) and balance stress

$$
\text{On y=h(x,t)} \qquad \frac{D}{Dt}(y-h(x,t))=0 \implies v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \qquad \text{(kinematic condition)},
$$

$$
\mathbf{t}=-p_a\mathbf{n} \qquad \text{(stress balance)},
$$

where $p_a$ atmospheric/external pressure, $\mathbf{t}$ the traction vector and $\mathbf{n}$ outward unit normal given by

$$
\mathbf{t}=(t_1,t_2), \qquad t_i=\sigma_{ij}n_j, \qquad \mathbf{n}=(n_1,n_2)=\left(-\frac{\partial h}{\partial x},1\right)\left(1+\left(\frac{\partial h}{\partial x}\right)^2\right)^{-\frac{1}{2}}.
$$

### 3.4.3 Dimensionless problem

- Nondimensionalise:

$$
x=L\bar{x}, \quad y=\delta L\bar{y}, \quad h=\delta L\bar{h}, \quad u=U\bar{u}, \quad v=\delta U\bar{v}, \quad t=\frac{L}{U}\bar{t},
$$

$$
p=\underbrace{\frac{\mu U}{\delta^2 L}}_{\substack{\text{viscous}\\ \text{scaling}}}\bar{p}, \quad \sigma_{ij}=\frac{\mu U}{L}\bar{\sigma}_{ij}, \quad p_a=\frac{\mu U}{\delta^2 L}\bar{p}_a.
$$

- Dimensionless governing equations (introducing usual Reynolds number$Re=\rho UL/\mu$):

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}} &= 0\\
Re\delta^2\left(\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+\delta^2\frac{\partial^2 \bar{u}}{\partial \bar{x}^2},\\
Re\delta^4\left(\frac{\partial \bar{v}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) &= -\frac{\partial \bar{p}}{\partial \bar{y}}+\delta^2\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}+\delta^4\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}-\underbrace{\frac{\delta^3 L^2\rho g}{\mu U}}_{\text{Take } U=\delta^3L^2\rho g/\mu}.
\end{aligned}
$$

Note that $U$ is not fixed and we may choose it based on $g$ so that the dimensionless gravitational body force term becomes unity.

- Boundary conditions on the lower fixed surface remains unchanged:

$$
\text{On } \bar{y}=0 \qquad \bar{u}=\bar{v}=0.
$$

- On the free surface the kinematic condition remains unchanged:

$$
\text{On } \bar{y}=\bar{h}(\bar{x},\bar{t}) \qquad \bar{v}=\frac{\partial \bar{h}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{h}}{\partial \bar{x}}.
$$

- On the free surface the stress balance condition can be simplified:

$$\bar{\sigma}_{11} = -\frac{\bar{p}}{\delta^2} + 2\frac{\partial \bar{u}}{\partial \bar{x}}, \qquad \bar{\sigma}_{12} = \frac{1}{\delta}\frac{\partial \bar{u}}{\partial \bar{y}} + \delta\frac{\partial \bar{v}}{\partial \bar{x}}, \qquad \bar{\sigma}_{22} = -\frac{\bar{p}}{\delta^2} + 2\frac{\partial \bar{v}}{\partial \bar{y}},$$

$$\bar{n}_1 = \delta\frac{d\bar{h}}{d\bar{x}}\left(1+\delta^2\left(\frac{d\bar{h}}{d\bar{x}}\right)^2\right)^{-\frac{1}{2}}, \qquad \bar{n}_2 = -\left(1+\delta^2\left(\frac{d\bar{h}}{d\bar{x}}\right)^2\right)^{-\frac{1}{2}}$$

$$\text{On } \bar{y} = \bar{h}(\bar{x},\bar{t}) \qquad \bar{t}_2 = \bar{\sigma}_{21}\bar{n}_1 + \bar{\sigma}_{22}\bar{n}_2 = -\frac{\bar{p}_a}{\delta^2}\bar{n}_2 \implies \bar{p} = \bar{p}_a + O(\delta^2),$$

$$\bar{t}_1 = \bar{\sigma}_{11}\bar{n}_1 + \bar{\sigma}_{12}\bar{n}_2 = -\frac{\bar{p}_a}{\delta^2}\bar{n}_1 \implies \frac{\partial \bar{u}}{\partial \bar{y}} + O(\delta^2) = 0.$$

### 3.4.4 Lubrication approximation

- Assuming $\delta \ll 1$ and $Re\delta^2 \ll 1$, at leading order (on dropping bars) gives the lubrication model:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{87}$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{88}$$

$$0 = -\frac{\partial p}{\partial y} - 1, \tag{89}$$

with boundary conditions

$$\text{on } y = 0: \qquad u = v = 0, \tag{90}$$

$$\text{on } y = h(x,t): \qquad v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}, \quad p = p_a, \quad \frac{\partial u}{\partial y} = 0. \tag{91}$$

- Integrating (89) and using (91) gives

$$p = p_a + h - y. \tag{92}$$

- Integrating (88) and using (90) and (91) gives

$$u = \frac{1}{2}\frac{\partial p}{\partial x}y(y-2h).$$

- Integrating (87) across the layer:

$$\int_0^{h(x,t)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dy = 0 \implies \int_0^{h(x,t)} \frac{\partial u}{\partial x}\, dy + v(x,h(x,t)) - v(x,0) = 0.$$

Using the kinematic condition from (91) and boundary condition in (90) gives

$$\int_0^{h(x,t)} \frac{\partial u}{\partial x}\, dy + u(x,h)\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} = 0 \underbrace{\implies}_{\text{Leibniz's rule}} \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,y)\, dy + \frac{\partial h}{\partial t} = 0$$

Using the expression for $u$ gives

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right).$$

### 3.4.5 Thin-film equation for $h(x,t)$

- Using the pressure expression (92) gives

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right).$$

- This is an evolution equation in time (and second order in space) to determine the free-surface $h(x,t)$.
- It is the thin-film equation for gravity driven flow and also termed the porous medium equation.
- It is a nonlinear version of the heat equation.
- Coefficient of highest derivative (the mobility $h^3/3$) vanishes when $h=0$, so the partial differential equation is degenerate at $h=0$.
- Two important manifestations of this degeneracy are the existence of compactly supported mass preserving similarity solutions and "waiting time" behaviour
- Higher order versions are obtained for different driving forces:

$$\text{Surface tension: } p = \underbrace{-\frac{\partial^2 h}{\partial x^2}}_{\substack{\text{surface}\\ \text{curvature}}} \qquad \frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial^3 h}{\partial x^3}\right) \qquad \text{(fourth-order).}$$

$$\text{Light plate: } p = \underbrace{\frac{\partial^4 h}{\partial x^4}}_{\substack{\text{Euler-Bernoulli}\\ \text{beam theory}}} \qquad \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial^5 h}{\partial x^5}\right) \qquad \text{(sixth-order).}$$

# Appendix

## A Lubrication Theory and Thin Films

### A.1 General Lubrication Theory

#### A.1.1 Motivation

- How can a thin fluid layer support a large normal load?
- Example: sheet of paper sliding across a table.
- Lubrication theory is the analysis of flows in thin layers whose aspect ratio (depth / length) is small.
- Lubrication theory combines ideas from slow flow and boundary layer theory to simplify the Navier-Stokes equations.
- Use the same methodology as for unidirectional flows: derive a general version of the simplified equation, then apply it to several configurations that are of practical importance, namely flow in a slider bearing, squeeze film and Hele-Shaw cell.

#### A.1.2 Governing equations

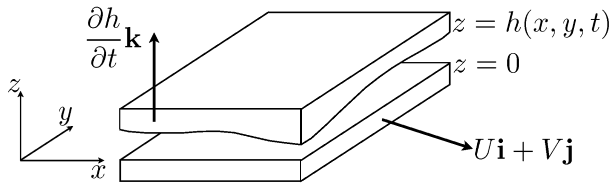
- Begin with the incompressible Navier-Stokes equations

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{u}, \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0,$$

  where $\rho$ is the density, $\mu$ is the viscosity, $\mathbf{u}(x,\ y,\ z,\ t) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the velocity, $p(x,\ y,\ z,\ t)$ is the pressure and the convective derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

- Consider channel flow between rigid boundaries at $z = 0$ and $z = h(x,\ y,\ t)$ , where $h(x,\ y,\ t)$ is the prescribed channel thickness:




- Lower boundary has velocity $U\mathbf{i} + V\mathbf{j}$, so no-slip and no-flux boundary conditions are

$$u = U,\ v = V,\ w = 0 \quad \text{ on } z = 0.$$

- Assuming the upper boundary has velocity $\frac{\partial h}{\partial t}\mathbf{k}$ (*i.e.* that it moves in the vertical direction only), the no-slip and no-flux boundary conditions are

$$u = 0,\ v = 0,\ w = \frac{\partial h}{\partial t} \quad \text{ on } z = h(x,\ y,\ t).$$

- We will prescribe appropriate boundary conditions at channel sides later on (after reduction of dimensionality).

### A.1.3 Nondimensionalization

- Let $U_0$ be a typical horizontal flow speed, *i.e.*

$$[u],\ [v],\ [U],\ [V] = U_0,$$

where [ ] denotes here the typical size of the bracketed dimensional quantity.

- Let $L$ and $\delta L$ be typical horizontal and vertical length scales, respectively, *i.e.*

$$[x],\ [y] = L; \qquad [z],\ [h] = \delta L,$$

where the aspect ratio $\delta \ll 1$.

- Continuity equation: $\dfrac{\partial w}{\partial z} = -\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}$

$$\Rightarrow \quad \frac{[w]}{[z]} = \frac{[u]}{[x]} \quad \Rightarrow \quad \frac{[w]}{\delta L} = \frac{U_0}{L} \quad \Rightarrow \quad [w] = \delta U_0.$$

- Hence nondimensionalize by scaling

$$\begin{aligned} x &= L\hat{x}, \quad y = L\hat{y}, \quad z = \delta L\hat{z}, \\ u &= U_0\hat{u}, \quad v = U_0\hat{v}, \quad w = \delta U_0\hat{w}, \\ h &= \delta L\hat{h}, \quad t = \frac{L}{U_0}\hat{t}, \quad p = p_{atm} + [p]\hat{p}, \end{aligned}$$

where $\delta$ is small ($i.e.$ $\delta \ll 1$) , $p_{atm}$ is atmospheric pressure and the scale of pressure variations $[p]$ is to be determined.

### A.1.4 The reduced Reynolds number

- The convective derivative becomes

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{1}{L/U_0}\left(\frac{\partial}{\partial \hat{t}} + \hat{u}\frac{\partial}{\partial \hat{x}} + \hat{v}\frac{\partial}{\partial \hat{y}} + \hat{w}\frac{\partial}{\partial \hat{z}}\right) \equiv \frac{1}{L/U_0}\frac{\mathrm{D}}{\mathrm{D}\hat{t}}.$$

- Hence the $x$-momentum equation becomes

$$\frac{\rho U_0}{L/U_0}\frac{\mathrm{D}\hat{u}}{\mathrm{D}\hat{t}} = -\frac{[p]}{L}\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\mu U_0}{L^2}\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{1}{\delta^2}\frac{\partial^2 \hat{u}}{\partial \hat{z}^2}\right)$$

- Dividing through by $\mu U_0/\delta^2 L^2$ gives

$$\delta^2 Re \frac{\mathrm{D}\hat{u}}{\mathrm{D}\hat{t}} = -\frac{\delta^2 L[p]}{\mu U_0}\frac{\partial \hat{p}}{\partial \hat{x}} + \delta^2\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \delta^2\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2},$$

where

$$Re = \frac{\rho L U_0}{\mu} = \frac{L U_0}{\nu} = \frac{[x][u]}{\nu}$$

is the Reynolds number based on the length and velocity scale in the $x$-direction.

- For $\delta \ll 1$, the key parameter in the $x$-momentum equation is the reduced Reynolds number

$$\delta^2 Re = \frac{(\delta L)(\delta U_0)}{\nu} = \frac{[z][w]}{\nu},$$

*i.e.* the Reynolds number based on the length and velocity scale in the vertical direction.

- In lubrication theory, we assume the reduced Reynolds number is small, *i.e.*

$$\delta^2 Re \ll 1,$$

so that inertial terms are much smaller than viscous terms; note that $Re$ need not be small!

- Compare this with boundary layer theory in §3, where we set the reduced Reynolds number equal to unity in the boundary layer, giving the boundary layer thickness $\delta = Re^{-1/2}$ for $Re \gg 1$.
- Finally, we balance viscous and pressure terms (to avoid a triviality) by setting

$$\frac{\delta^2 L[p]}{\mu U_0} = 1 \quad \Rightarrow \quad [p] = \frac{\mu U_0}{\delta^2 L}.$$

- Hence the pressure scale in lubrication theory is a factor $1/\delta^2$ larger than the pressure scale in Stokes flow.

### A.1.5 Dimensionless problem

- Substitute the scalings into the Navier-Stokes equations and simplify them to obtain

$$\begin{aligned}
\delta^2 Re \frac{\mathrm{D}\hat{u}}{\mathrm{D}\hat{t}} &= -\frac{\partial \hat{p}}{\partial \hat{x}} + \delta^2 \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \delta^2 \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2}, \\
\delta^2 Re \frac{\mathrm{D}\hat{v}}{\mathrm{D}\hat{t}} &= -\frac{\partial \hat{p}}{\partial \hat{y}} + \delta^2 \frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \delta^2 \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{z}^2}, \\
\delta^4 Re \frac{\mathrm{D}\hat{w}}{\mathrm{D}\hat{t}} &= -\frac{\partial \hat{p}}{\partial \hat{z}} + \delta^4 \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \delta^4 \frac{\partial^2 \hat{w}}{\partial \hat{y}^2} + \delta^2 \frac{\partial^2 \hat{w}}{\partial \hat{z}^2}, \\
0 &= \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}},
\end{aligned}$$

where the Reynolds number $Re = \rho L U_0/\mu$.

- Scaling $U = U_0\hat{U}$ and $V = U_0\hat{V}$, the no-flux and no-slip boundary conditions become

$$\begin{aligned}
\hat{u} = \hat{U},\ \hat{v} = \hat{V},\ \hat{w} = 0 \quad &\text{on } \hat{z} = 0, \\
\hat{u} = 0,\ \hat{v} = 0,\ \hat{w} = \frac{\partial \hat{h}}{\partial \hat{t}} \quad &\text{on } \hat{z} = \hat{h}(\hat{x}, \hat{y}, \hat{t}).
\end{aligned}$$

### A.1.6 Lubrication approximation

- Assume that both the aspect ratio $\delta$ and the reduced Reynolds number $\delta^2 Re$ are small, *i.e.*

$$\delta,\ \delta^2 Re \ll 1,$$

to obtain at leading order the quasi-steady lubrication equations (dropping the hats ˆ):

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2}, \tag{93}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2}, \tag{94}$$

$$0 = -\frac{\partial p}{\partial z}, \tag{95}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{96}$$

with boundary conditions

$$\begin{aligned} u &= U,\ v = V,\ w = 0 \quad \text{on } z = 0, \\ u &= 0,\ v = 0,\ w = \frac{\partial h}{\partial t} \quad \text{on } z = h(x,\ y,\ t). \end{aligned} \tag{97}$$

### A.1.7 Cross-layer averaged conservation of mass expression

- Integrate the incompressibility condition (96) from $z = 0$ to $z = h$:

$$0 = \int_0^h \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \mathrm{d}z = \frac{\partial}{\partial x}\left(\int_0^h u \mathrm{d}z\right) - u \left.\frac{\partial h}{\partial x}\right|_{z=h} + \frac{\partial}{\partial y}\left(\int_0^h v \mathrm{d}z\right) - v \left.\frac{\partial h}{\partial y}\right|_{z=h} + [w]_{z=0}^{z=h},$$

  by Leibniz's integral rule.

- Applying the kinematic boundary conditions in (97), we obtain the cross-layer-averaged conservation of mass expression

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\overline{u}) + \frac{\partial}{\partial y}(h\overline{v}) = 0.$$

  where the mean velocities are given by

$$\overline{u} := \frac{1}{h}\int_0^h u \mathrm{d}z, \qquad \overline{v} := \frac{1}{h}\int_0^h v \mathrm{d}z,$$

### A.1.8 Integrating the momentum equations

- To determine the mean velocities $\overline{u}$ and $\overline{v}$, need to integrate the momentum equations to obtain $u$ and $v$.
- This is possible because the $z$-momentum equation (95) implies that at leading order the pressure is independent of $z$.
- Hence, for fixed $(x,\ y)$ , the $x$- and $y$-momentum equations (93)–(94) are ordinary differential equations in $z$ for $u$ and $v$, giving after two integrations the expressions

$$u = \frac{1}{2}\frac{\partial p}{\partial x} z^2 + A_1(x,\ y,\ t)z + B_1(x,\ y,\ t), \qquad v = \frac{1}{2}\frac{\partial p}{\partial y} z^2 + A_2(x,\ y,\ t)z + B_2(x,\ y,\ t)\ ,$$

  where $A_i,\ B_i$ are arbitrary functions of $(x,\ y,\ t)$ .

- Determine $A_i,\ B_i$ by applying the no-slip boundary conditions in (97) to obtain

$$u = -\frac{1}{2}\frac{\partial p}{\partial x} z(h - z) + U\left(1 - \frac{z}{h}\right), \qquad v = -\frac{1}{2}\frac{\partial p}{\partial y} z(h - z) + V\left(1 - \frac{z}{h}\right)\ ,$$

  *i.e.* Poiseuille / Couette flow in each direction.

- Integrate again to obtain mean velocities

$$\overline{u} = \frac{1}{h}\int_0^h u \mathrm{d}z = -\frac{h^2}{12}\frac{\partial p}{\partial x} + \frac{U}{2}, \qquad \overline{v} = \frac{1}{h}\int_0^h v \mathrm{d}z = -\frac{h^2}{12}\frac{\partial p}{\partial y} + \frac{V}{2}.$$

### A.1.9 Reynolds lubrication equation

- In summary, we have obtained the cross-layer averaged conservation of mass expression

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(h\overline{u}\right)+\frac{\partial}{\partial y}\left(h\overline{v}\right)=0, \tag{98}$$

where the mean velocities

$$\overline{u}=-\frac{h^2}{12}\frac{\partial p}{\partial x}+\frac{U}{2}, \qquad \overline{v}=-\frac{h^2}{12}\frac{\partial p}{\partial y}+\frac{V}{2}. \tag{99}$$

- Substituting for $\overline{u},\overline{v}$, we obtain a version of Reynolds lubrication equation

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12}\frac{\partial p}{\partial y}\right)=\frac{\partial h}{\partial t}+\frac{U}{2}\frac{\partial h}{\partial x}+\frac{V}{2}\frac{\partial h}{\partial y}, \tag{100}$$

which is an elliptic equation for the pressure $p(x,\ y,\ t)$ , since the thickness $h(x,\ y,\ t)$ is prescribed (*cf.* the thin-film equations in §A.4).

- Use Reynolds lubrication equation (100) to study flow in a slider bearing, squeeze film and Hele-Shaw cell.

### A.1.10 Stress tensor

- Nondimensionalize the stress tensor $\sigma_{ij}$ and stress vector $\mathbf{t}$ by scaling them with the pressure scale $[p]$, *i.e.* set

$$\sigma_{ij}=\frac{\mu U_0}{\delta^2 L}\hat{\sigma}_{ij}, \qquad \mathbf{t}=\frac{\mu U_0}{\delta^2 L}\hat{\mathbf{t}}$$

and then drop immediately the hats ˆ denoting dimensionless variables.

- In the lubrication regime, the dimensionless stress tensor is given by

$$\begin{aligned}
\sigma_{11} &= -p+2\delta^2\frac{\partial u}{\partial x}\sim -p,\\
\sigma_{22} &= -p+2\delta^2\frac{\partial v}{\partial y}\sim -p,\\
\sigma_{33} &= -p+2\delta^2\frac{\partial w}{\partial z}\sim -p.\\
\sigma_{12} &= \delta^2\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}\sim \delta^2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
\sigma_{13} &= \delta\frac{\partial u}{\partial z}+\delta^3\frac{\partial w}{\partial x}\sim \delta\frac{\partial u}{\partial z},\\
\sigma_{23} &= \delta\frac{\partial v}{\partial z}+\delta^3\frac{\partial w}{\partial y}\sim \delta\frac{\partial v}{\partial z}.
\end{aligned}$$

- Dimensionless stress on the upper wall is given by

$$\mathbf{t}(\mathbf{n})=\mathbf{e}_i\sigma_{ij}n_j,$$

where

$$\mathbf{n}=n_j\mathbf{e}_j=\frac{\delta h_x\mathbf{e}_1+\delta h_y\mathbf{e}_2-\mathbf{e}_3}{(1+(\delta h_x)^2+(\delta h_y)^2)^{1/2}}$$

is the downward unit normal to the upper boundary.

- For $\delta\ll 1$, obtain at leading order

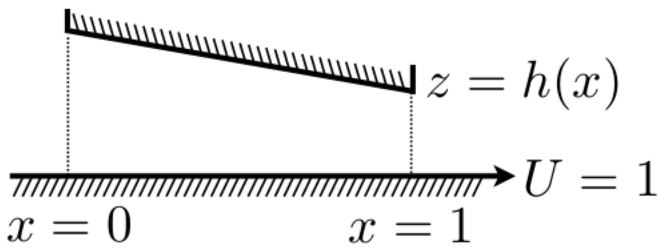
$$\begin{aligned}
\mathbf{t}(\mathbf{n}) &\sim -\delta\left(p\frac{\partial h}{\partial x}+\left.\frac{\partial u}{\partial z}\right|_{z=h}\right)\mathbf{e}_1-\delta\left(p\frac{\partial h}{\partial y}+\left.\frac{\partial v}{\partial z}\right|_{z=h}\right)\mathbf{e}_2+p\mathbf{e}_3\\
&= -\delta\left(\frac{\partial(ph)}{\partial x}-\frac{1}{2}h\frac{\partial p}{\partial x}-\frac{U}{h}\right)\mathbf{e}_1-\delta\left(\frac{\partial(ph)}{\partial y}-\frac{1}{2}h\frac{\partial p}{\partial y}-\frac{V}{h}\right)\mathbf{e}_2+p\mathbf{e}_3.
\end{aligned} \tag{101}$$

- Hence, vertical component of stress is $O(1/\delta)$ larger than horizontal components, which explains why a thin viscous layer can support large normal loads, with relatively little horizontal resistance (*cf.* solid contact, where friction is proportional to the normal load).

## A.2 Examples

### A.2.1 Steady flow in a one-dimensional slider bearing

- Set $h = h(x)$ on $0 < x < 1$, with $U = 1,\ V = 0$:




- Reynolds lubrication equation (100) for the pressure $p = p(x)$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h^3}{12}\frac{\mathrm{d}p}{\mathrm{d}x}\right) = \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}x}.$$

- Typically ambient pressure at ends of bearing, so

$$p = 0 \quad \text{at } x = 0, 1.$$

- Integrate to obtain the pressure gradient

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{6(h - h_0)}{h^3},$$

  where the constant $h_0$ is to be determined.

- Note that $p(x)$ has a stationary point at $h = h_0$.

- Integrate again and apply boundary conditions to obtain the solution

$$p(x) = 6\int_0^x \frac{h(x) - h_0}{h(x)^3}\mathrm{d}x, \qquad \int_0^1 \frac{h(x) - h_0}{h(x)^3}\mathrm{d}x = 0.$$

- By (90), the dimensionless components of force on the slider in the $x$- and $z$-directions are given by

$$F_x = \delta\int_0^1 \frac{4h(x) - 3h_0}{h(x)^2}\mathrm{d}x, \qquad F_z = \int_0^1 p(x)dx.$$

- Note $F_x = O(\delta F_z)$ , so high loads, low friction.

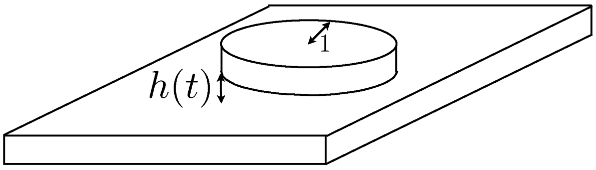
- Converging channel $h' < 0$, gives $p > 0$ and slider supports load.

- Diverging channel $h' > 0$, gives $p < 0$ and hence cavitation (*i.e.* gas bubble formation) if dimensional pressure becomes too small.

- Cavitation may lead to contact and wear: see *e.g.* Ockendon & Ockendon, Viscous flow, §4.1.

### A.2.2 Axisymmetric flow in a squeeze film

- Lower boundary stationary ($U = V = 0$) and upper boundary a flat circular unit disc at $z = h(t)$, $r < 1$:




- Seeking an axisymmetric solution $p = p(r,\ t)$, Reynolds lubrication equation (100) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rh^3}{12}\frac{\partial p}{\partial r}\right) = \frac{\mathrm{d}h}{\mathrm{d}t} \qquad \Rightarrow \qquad p(r,\ t) = \frac{3\dot{h}}{h^3}\left(r^2 + A(t)\ln r + B(t)\right),$$

  where $A(t)$ , $B(t)$ are arbitrary functions of $t$.

- Demand $p$ bounded and impose ambient pressure $p = 0$ at $r = 1$ to obtain

$$p(r,\ t) = \frac{3\dot{h}}{h^3}\left(r^2 - 1\right).$$

- By (101), the dimensionless upward force on the disc is given by

$$F_z = \int_0^{2\pi}\int_0^1 p(r,\ t) r\,\mathrm{d}r\,\mathrm{d}\theta = -\frac{3\pi\dot{h}}{2h^3}.$$

- If $\dot{h} > 0$, this force is negative (suction makes disc adhere to plate) and vice versa.

- If a constant load $N$ is applied to disc, then

$$Nt = -\int_0^t \frac{3\pi\dot{h}(\tau)}{2h(\tau)^3}\mathrm{d}\tau = \frac{3\pi}{4}\left(\frac{1}{h(t)^2} - \frac{1}{h(0)^2}\right),$$

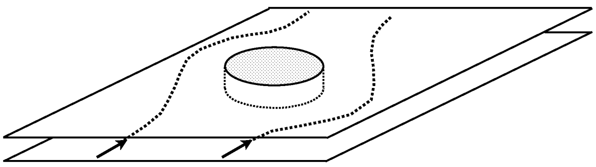
  so it takes infinite time to squeeze fluid out of the gap!

- In practice, this prediction is rendered invalid by neglected effects, *e.g.* surface roughness, fluid compressibility, plate curvature.

- Note that in the nondimensionalization, the typical plate velocity $W_0$ (say) determines the appropriate horizontal velocity scale $U_0 = W_0/\delta$ (by the continuity equation) and time scale $[t] = \delta L/W_0 = L/U_0$ (by the no-flux boundary condition on the plate).

### A.2.3 Flow in a Hele-Shaw cell

- Flow between two parallel stationary plates ($h$ constant, $U = V = 0$) driven by an external pressure gradient:



- The mean velocities (99) are given by

$$\overline{u} = -\frac{h^2}{12}\frac{\partial p}{\partial x} \qquad \overline{v} = -\frac{h^2}{12}\frac{\partial p}{\partial y},$$

  so Reynolds lubrication equation (100) reduces to Laplace's equation for the pressure $p(x,\ y,\ t)$ :

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0.$$

- Remarkably, since

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0, \qquad \frac{\partial \overline{u}}{\partial y} - \frac{\partial \overline{v}}{\partial x} = 0,$$

the mean flow (and in fact the flow at any fixed $z$) is the same as for two-dimensional incompressible inviscid irrotational flow with velocity potential $-h^2 p/12$.

- However, there are three important distinctions.

  (i) Although the streamlines are the same, the pressure is different (for an incompressible inviscid irrotational fluid flow, the pressure is given by Bernoulli's equation).

  (ii) The circulation $\Gamma$ around any closed curve $C$ lying in a horizontal plane is zero, whether or not the fluid domain is simply connected:

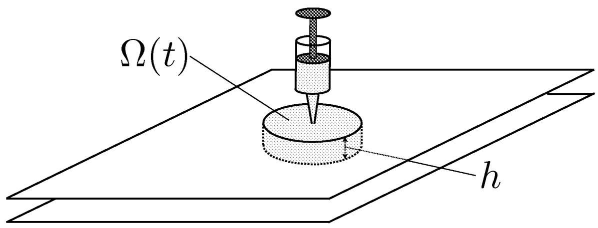
$$\Gamma = \int_C u\mathrm{d}x + v\mathrm{d}y = -\frac{1}{2}z(h-z)\int_C \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y = -\frac{1}{2}z(h-z)[p]_C = 0$$

  because $p$ is a single valued function of $x$ and $y$. Prediction: can use Hele-Shaw cell to visualize inviscid flows with zero circulation!

  (iii) Near a smooth obstacle the lubrication approximation breaks down in a boundary layer of horizontal width of the order of the gap thickness in which the cross-layer flow is governed at leading order by the quasi-two-dimensional Stokes flow equations (as the local Reynolds number $\delta^2 Re \ll 1$), so boundary layer structure completely different from Prandtl picture and no separation.

### A.2.4 Injection at a point in a Hele-Shaw cell

- A blob of fluid $\Omega(0)$ containing the origin lies within a Hele-Shaw cell ($h$ constant, $U = V = 0$) when at time $t = 0$ a source of constant strength $Q$ is introduced at the origin.




- This is a free boundary problem, as fluid domain $\Omega(t)$ must be determined as part of solution.

- Convenient to introduce vector notation

$$\overline{\mathbf{u}}(x,\ y,\ t) = \overline{u}\mathbf{i} + \overline{v}\mathbf{j}, \qquad \boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y},$$

so that Reynolds lubrication equation (100) and mean velocities (99) become

$$\boldsymbol{\nabla}\cdot(h\overline{\mathbf{u}}) = 0, \qquad \overline{\mathbf{u}} = -\frac{h^2}{12}\boldsymbol{\nabla} p.$$

- Since $h$ is constant, pressure $p(x,\ y,\ t)$ satisfies Laplace's equation

$$\boldsymbol{\nabla}^2 p = 0 \quad \text{in } \Omega(t).$$

- Source strength $Q$ at origin means

$$h\overline{\mathbf{u}} \sim \frac{Q}{2\pi r}\mathbf{e}_r \quad \text{as } r = \left(x^2+y^2\right)^{1/2} \to 0,$$

so that

$$\int_{r=\varepsilon} h\overline{\mathbf{u}}\cdot\mathbf{e}_r \mathrm{d}s \to Q \quad \text{as } \varepsilon \to 0.$$

- On free boundary $\partial\Omega(t)$ need to impose two boundary conditions to determine its location.
- Suppose free boundary is given by $f(x,\ y,\ t) = 0$ and has outward unit normal

$$\mathbf{n} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|}.$$

- The simplest boundary conditions are the zero pressure (neglecting surface tension) and kinematic conditions

$$p = 0,\ \frac{\mathrm{D}f}{\mathrm{D}t} = 0 \quad \text{on } f = 0,$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \boldsymbol{\nabla}.$$

- The kinematic condition says the outward normal velocity of the fluid is equal to that of the boundary, *i.e.*

$$\overline{\mathbf{u}} \cdot \mathbf{n} = \frac{\overline{\mathbf{u}} \cdot \boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = v_n \quad \text{on } \partial\Omega(t),$$

the outward normal velocity of the boundary being given by

$$v_n = -\frac{1}{|\boldsymbol{\nabla} f|}\frac{\partial f}{\partial t}\bigg|_{f=0}.$$

- Seek an axisymmetric solution in which

$$p = p(r,\ t), \quad \overline{\mathbf{u}} = -\frac{h^2}{12}\frac{\partial p}{\partial r}\mathbf{e}_r, \quad \Omega(t) = \{r < R(t)\}.$$

- Choose $f(x,\ y,\ t) = x^2 + y^2 - R(t)^2$, so that

$$\mathbf{n} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{2r} = \mathbf{e}_r,$$

then outward normal velocity of boundary

$$v_n = -\frac{1}{|\boldsymbol{\nabla} f|}\frac{\partial f}{\partial t}\bigg|_{f=0} = -\frac{-2R\dot{R}}{2r}\bigg|_{r=R} = \dot{R},$$

as expected.

- Hence, lubrication problem reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = 0 \quad \text{for } r < R(t),$$

$$-\frac{h^2}{12}\frac{\partial p}{\partial r} \sim \frac{Q}{2\pi h r} \quad \text{as } r \to 0,$$

$$p = 0,\ -\frac{h^2}{12}\frac{\partial p}{\partial r} = \dot{R} \quad \text{on } r = R(t).$$

- Integrate and apply the boundary condition at the origin to obtain

$$-\frac{h^2}{12}\frac{\partial p}{\partial r} = \frac{Q}{2\pi h r}.$$

- Integrate again and apply $p(R(t),\ t) = 0$ to obtain

$$p(r,\ t) = -\frac{6Q}{\pi h^3}\log\left(\frac{r}{R(t)}\right).$$

- Finally, apply kinematic condition to obtain

$$\left.\frac{Q}{2\pi h r}\right|_{r=R} = \dot{R} \quad \Rightarrow \quad \pi h R(t)^2 = \pi h R(0)^2 + Qt,$$

  which represents global conservation of mass.

- Prediction: flow reversible upon changing sign of $Q$.

### A.2.5 Squeeze film with a free boundary

- A blob of fluid lies between two parallel plates whose separation is $h(t)$ (so $h = h(t)$ and $U = V = 0$).
- Similar to injection problem, except flow driven by plate motion.
- Seeking an axisymmetric solution $p = p(r,\ t)$ with a circular blob of radius $R(t)$ results in the problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{h^3 r}{12}\frac{\partial p}{\partial r}\right) = \dot{h} \qquad \text{for } r < R(t),$$

$$p \text{ bounded} \qquad \text{as } r \to 0,$$

$$p = 0,\ -\frac{h^2}{12}\frac{\partial p}{\partial r} = \dot{R} \qquad \text{on } r = R(t).$$

- The solution is

$$p(r,\ t) = \frac{3\dot{h}(t)}{h(t)^3}\left(r^2 - R(t)^2\right),$$

  with $\pi h(t) R(t)^2 = \pi h(0) R(0)^2$, representing global conservation of mass.

- Prediction: flow reversible upon reversing path of $h(t)$ .

## A.3 The Saffman-Taylor instability and viscous fingering

- For the examples in §A.2.4 and §A.2.5, experiments show that the flow is not time reversible: when the interface is advancing small perturbations to the interface shrink, so the circular base state is stable; however, when the interface is retreating small perturbations grow, so the circular base state is unstable.
- To analyze this phenomena we use linear stability theory to study the local problem near the interface: consider a Hele-Shaw cell consisting of a region of saturated flow in $y < H(x,\ t)$ which is separated from a dry region in $y > H(x,\ t)$ by an interface at $y = H(x,\ t)$.
- In the dimensionless variables of §A.2.3, with $h^2/12 = 1$ for brevity, the flow is governed by

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0, \quad \overline{u} = -\frac{\partial p}{\partial x}, \quad \overline{v} = -\frac{\partial p}{\partial y}.$$

- Impose the zero-pressure and kinematic conditions

$$p = 0,\ \overline{v} = \frac{\partial H}{\partial t} + \overline{u}\frac{\partial H}{\partial x} \qquad \text{on } y = H(x,\ t).$$

- Take the base state to be the travelling wave solution moving upward with constant speed $V$, *i.e.*

$$\overline{u} = 0, \quad \overline{v} = V, \quad H = Vt, \quad p = p_0(y,\ t) = -V(y - Vt),$$

so that the flow is driven by the prescribed pressure gradient

$$\frac{\partial p}{\partial y} \to -V \quad \text{as } y \to -\infty.$$

- The key step is to write the evolution problem relative to the base state by changing to a travelling wave frame $(x,\ \eta)$ moving with the interface by setting

$$\begin{aligned} y - Vt &= \eta, \\ \overline{u} &= \varepsilon \hat{u}(x,\ \eta,\ t), \\ \overline{v} - V &= \varepsilon \hat{v}(x,\ \eta,\ t), \\ p - p_0(y,\ t) &= \varepsilon \hat{p}(x,\ \eta,\ t) \\ H - Vt &= \varepsilon \hat{H}(x,\ t), \end{aligned}$$

where $\varepsilon$ is an arbitrary small parameter.

- Use the chain rule to derive the new problem

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial \eta} = 0 \quad \text{in } \eta < \varepsilon \hat{H}(x,\ t), \tag{102}$$

with

$$\hat{p} = V\hat{H},\ \hat{v} = \frac{\partial \hat{H}}{\partial t} + \varepsilon \hat{u} \frac{\partial \hat{H}}{\partial x} \quad \text{on } \eta = \varepsilon \hat{H}(x,\ t), \tag{103}$$

and

$$\hat{p} = \mathrm{o}(-\eta) \quad \text{as } \eta \to -\infty, \tag{104}$$

where

$$\hat{u} = -\frac{\partial \hat{p}}{\partial x}, \qquad \hat{v} = -\frac{\partial \hat{p}}{\partial \eta}. \tag{105}$$

- As $\varepsilon \ll 1$, can now proceed as for small amplitude water waves, as follows.

  (i) Linearize the boundary conditions (103) and impose them on $\eta = 0$, so that at leading order (102)–(103) become

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial \eta} = 0 \quad \text{in } \eta < 0, \tag{106}$$

  with

$$\hat{p} = V\hat{H},\ \hat{v} = \frac{\partial \hat{H}}{\partial t} \quad \text{on } \eta = 0. \tag{107}$$

  Hence, by (104)–(107), $\hat{p}(x,\ \eta,\ t)$ satisfies Laplace's equation

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial \eta^2} = 0 \quad \text{in } \eta < 0, \tag{108}$$

  with boundary conditions

$$\hat{p} = V\hat{H},\ \frac{\partial \hat{p}}{\partial \eta} = -\frac{\partial \hat{H}}{\partial t} \quad \text{on } \eta = 0 \tag{109}$$

  and

$$\hat{p} = \mathrm{o}(-\eta) \quad \text{as } \eta \to -\infty. \tag{110}$$

(ii) Seek a separable solution to (108)–(110) of the form

$$\hat{p}(x,\ \eta,\ t) = \text{Re}\left(g(\eta)\text{e}^{ikx+\lambda t}\right), \qquad \hat{H}(x,\ t) = \text{Re}\left(\text{e}^{ikx+\lambda t}\right),$$

where $k$ is the (real) wavenumber, $\text{Re}(\lambda(k))$ is the growth rate and Re means 'real part'. Deduce from (108)–(110) that $g(\eta)$ satisfies the second-order linear ordinary differential equation

$$g'' - k^2 g = 0, \tag{111}$$

with boundary conditions

$$g(0) = V,\ g'(0) = -\lambda; \qquad g(-\infty) \text{ bounded.} \tag{112}$$

(iii) The problem (111)–(112)is an eigenvalue problem for $\lambda(k)$: there is a nontrivial solution for $g(\eta)$ if and only if

$$\lambda = -V\,|k|\,. \tag{113}$$

Determine the region of instability from the dispersion relation (113):

(a) $V > 0 \neq k \ \Rightarrow\ \lambda < 0\ \Rightarrow$ perturbations decay, so interface linearly stable.

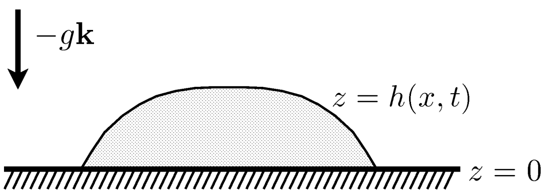
(b) $V < 0 \neq k \ \Rightarrow\ \lambda > 0\ \Rightarrow$ perturbations grow, so interface linearly unstable.

- This is an example of the Saffman-Taylor instability: the interface between two immiscible fluids is unstable when it moves toward the more viscous fluid.
- In the unstable regime experiments show that small perturbations grow into "viscous fingers" that evolve in a highly nonlinear manner sensitive to the driving force, surface tension and substrate properties.
- This instability causes significant problems in *e.g.* the oil extraction industry because the flow of oil and water through a porous media (such as rock) is governed by a similar set of differential equations (the Darcy flow model: see *e.g.* Ockendon & Ockendon, chapter 5).

## A.4 Thin films

### A.4.1 Motivation

- Consider gravity-driven flow of a thin two-dimensional viscous layer of *a priori* unknown thickness $h(x,\ t)$ on a stationary horizontal plate at $z = 0$ below a vacuum:




- Such a thin film may model spreading of a raindrop, molten lava or an ice sheet.
- This is a free boundary problem, as fluid domain $0 < z < h(x,\ t)$ must be determined as part of solution.
- **New features:** need to incorporate gravity into the incompressible Navier-Stokes equations and prescribe appropriate free boundary conditions at the free surface $z = h(x,\ t)$ .
- **Key idea:** assume aspect ratio (depth / length) is small and apply lubrication theory.

### A.4.2 Dimensional problem

- The flow is governed by the incompressible Navier-Stokes equations (22)-(23) with $\mathbf{F} = -g\mathbf{k}$, so that

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \mu\boldsymbol{\nabla}^2\mathbf{u} - \rho g\mathbf{k}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0,$$

where $\rho$ is the density, $\mu$ is the viscosity, $g$ is the acceleration due to gravity, $\mathbf{u}(x,\ z,\ t) = u\mathbf{i} + w\mathbf{k}$ is the velocity, $p(x,\ z,\ t)$ is the pressure and the convective derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}.$$

- On the substrate impose the no-flux and no-slip boundary conditions

$$u = 0,\ w = 0 \quad \text{on } z = 0.$$

- On the free surface $z = h(x,\ t)$ need to impose three boundary conditions to determine its location: the simplest are

$$\text{no-flux (1 scalar boundary condition):} \qquad w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x},$$

$$\text{zero-stress (2 scalar boundary conditions):} \quad \mathbf{t}(\mathbf{n}) = 0,$$

where $\mathbf{t}(\mathbf{n}) = \mathbf{e}_i\sigma_{ij}n_j$ is the stress exerted on the free surface by fluid and $\mathbf{n}$ is the downward pointing unit normal, *i.e.*

$$\mathbf{n} = \frac{h_x\mathbf{e}_1 - \mathbf{e}_3}{(1+h_x^2)^{1/2}}.$$

### A.4.3 Nondimensionalization

- We make the usual lubrication scalings:

$$\begin{aligned}
x &= L\hat{x} && (\ L = [x] \text{ prescribed initially}),\\
h &= \delta L\hat{h} && (\ \delta = [h]/[x] \text{ prescribed initially}),\\
z &= \delta L\hat{z} && ([\mathrm{z}] = [\mathrm{h}])\\
u &= U\hat{u} && (\ U = [u] \text{ to be determined}),\\
w &= \delta U\hat{w} && ([w]/[z] = [u]/[x] \text{ by cty. eq.}),\\
t &= \frac{L}{U}\hat{t}, && (\ [h]/[t] = [w] \text{ by kinematic boundary condition}),\\
p &= \frac{\mu U}{\delta^2 L}\hat{p} && ([p]/[x] = \mu[u]/[z]^2 \text{ by x-mom. eq. }),\\
\sigma_{ij} &= \frac{\mu U}{\delta^2 L}\hat{\sigma}_{ij} && ([\sigma_{ij}] = [p]),\\
\mathbf{t} &= \frac{\mu U}{\delta^2 L}\hat{\mathbf{t}} && ([\mathbf{t}] = [p]).
\end{aligned}$$

### A.4.4 Dimensionless problem

- Substitute scalings into the two-dimensional Navier-Stokes equations and simplify to obtain

$$\delta^2 Re\frac{\mathrm{D}\hat{u}}{\mathrm{D}\hat{t}} = -\frac{\partial\hat{p}}{\partial\hat{x}} + \delta^2\frac{\partial^2\hat{u}}{\partial\hat{x}^2} + \frac{\partial^2\hat{u}}{\partial\hat{z}^2},$$

$$\delta^4 Re \frac{\mathrm{D}\hat{w}}{\mathrm{D}\hat{t}} = -\frac{\partial \hat{p}}{\partial \hat{z}} + \delta^4 \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \delta^2 \frac{\partial^2 \hat{w}}{\partial \hat{z}^2} - \frac{\delta^3 \rho g L^2}{\mu U},$$

$$0 = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{w}}{\partial \hat{z}},$$

where the Reynolds number $Re = \rho L U/\mu$ and dimensionless convective derivative

$$\frac{\mathrm{D}}{\mathrm{D}\hat{t}} = \frac{\partial}{\partial \hat{t}} + \hat{u}\frac{\partial}{\partial \hat{x}} + \hat{w}\frac{\partial}{\partial \hat{z}}.$$

- Key step: assume gravity drives flow by setting

$$U = \frac{\delta^3 \rho g L^2}{\mu}.$$

- On the substrate the no-flux and no-slip boundary conditions are unchanged:

$$\hat{u} = 0, \ \hat{w} = 0 \quad \text{on } \hat{z} = 0.$$

- On the free surface the kinematic condition is unchanged

$$\hat{w} = \frac{\partial \hat{h}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{h}}{\partial \hat{x}} \quad \text{on } \hat{z} = \hat{h}(\hat{x}, \hat{t}).$$

- The zero-stress condition becomes

$$\hat{\mathbf{t}}(\mathbf{n}) = 0 \quad \text{on } \hat{z} = \hat{h}(\hat{x}, \hat{t}),$$

  where

$$\hat{\mathbf{t}}(\mathbf{n}) = \mathbf{e}_1(\hat{\sigma}_{11}n_1 + \hat{\sigma}_{13}n_3) + \mathbf{e}_3(\hat{\sigma}_{31}n_1 + \hat{\sigma}_{33}n_3),$$

  with

$$\hat{\sigma}_{11} = -\hat{p} + 2\delta^2\frac{\partial \hat{u}}{\partial \hat{x}}, \quad \hat{\sigma}_{13} = \hat{\sigma}_{31} = \delta\frac{\partial \hat{u}}{\partial \hat{z}} + \delta^3\frac{\partial \hat{w}}{\partial \hat{x}}, \quad \hat{\sigma}_{33} = -\hat{p} + 2\delta^2\frac{\partial \hat{w}}{\partial \hat{z}},$$

  and

$$n_1 = \frac{\delta \hat{h}_{\hat{x}}}{\left(1 + (\delta \hat{h}_{\hat{x}})^2\right)^{1/2}}, \qquad n_3 = -\frac{1}{\left(1 + (\delta \hat{h}_{\hat{x}})^2\right)^{1/2}}.$$

- Hence, the stress vector

$$\hat{\mathbf{t}}(\mathbf{n}) \sim \left[-\delta\left(\hat{p}\frac{\partial \hat{h}}{\partial \hat{x}} + \frac{\partial \hat{u}}{\partial \hat{z}}\right)\mathbf{e}_1 + \hat{p}\mathbf{e}_3\right]_{\hat{z}=\hat{h}} \qquad \text{as } \delta \to 0.$$

### A.4.5 Lubrication approximation

- Assume that both the aspect ratio $\delta$ and the reduced Reynolds number $\delta^2 Re$ are small, *i.e.*

$$\delta, \ \delta^2 Re = \frac{\delta^5 g L^3}{\nu^2} \ll 1,$$

  to obtain at leading order the quasi-steady lubrication equations (dropping the hats ˆ):

$$0 = -\frac{\partial p}{x} + \frac{\partial^2 u}{\partial z^2} \tag{114}$$

$$0 = -\frac{\partial p}{\partial z} - 1, \tag{115}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}, \tag{116}$$

- On the substrate, the no-flux and no-slip boundary conditions are unchanged:

$$u = 0,\ w = 0 \quad \text{on } z = 0. \tag{117}$$

- On the free surface, the kinematic and zero-stress boundary conditions become

$$\left.\begin{array}{rcll} w &=& \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} & \text{(no flux)}, \\ p &=& 0 & \text{(zero normal stress)} \\ \dfrac{\partial u}{\partial z} &=& 0 & \text{(zero shear stress)} \end{array}\right\} \quad \text{on } z = h(x,\ t). \tag{118}$$

### A.4.6 Cross-layer averaged conservation of mass expression

- Integrate the incompressibility condition (116) from $z = 0$ to $z = h$:

$$0 = \int_0^h \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\mathrm{d}z = \frac{\partial}{\partial x}\left(\int_0^h u\mathrm{d}z\right) - u\frac{\partial h}{\partial x}\bigg|_{z=h} + \big[w\big]_{z=0}^{z=h},$$

  by Leibniz's integral rule.

- Applying the no-flux boundary conditions on $z = 0,\ h$ in (117)–(118), we obtain

$$0 = \frac{\partial}{\partial x}\left(\int_0^h u\mathrm{d}z\right) + \frac{\partial h}{\partial t}.$$

- Hence, defining the mean $x$-velocity by

$$\overline{u} := \frac{1}{h}\int_0^h u\mathrm{d}z, \tag{119}$$

  we obtain the cross-layer averaged conservation of mass expression

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\overline{u}) = 0. \tag{120}$$

### A.4.7 Integrating the momentum equations

- To determine the mean velocity $\overline{u}$, need to integrate the momentum equations to obtain $u$.

- This is possible because $z$-momentum equation (115) implies that

$$p = h(x,\ t) - z$$

  by the normal stress condition in (118).

- At leading order the pressure is hydrostatic, receiving a contribution from weight of fluid above only.

- For fixed $(x,\ t)$ , $x$-momentum equation (114) is an ordinary differential equation in $z$ for $u$, subject to the no-slip condition in (117) and the zero-shear-stress condition in (118), giving after two integrations the expression

$$u = \frac{\partial h}{\partial x}\left(\frac{1}{2}z^2 - hz\right)$$

- By (119), the mean velocity

$$\overline{u} = \frac{1}{h}\int_0^h \frac{\partial h}{\partial x}\left(\frac{1}{2}z^2 - hz\right)\mathrm{d}z = -\frac{h^2}{3}\frac{\partial h}{\partial x}.$$

### A.4.8 Thin-film equation for $h(x,\ t)$

- Hence, the cross-layer averaged conservation of mass expression (120) becomes

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\left(h\overline{u}\right) = \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right),$$

  which is a nonlinear version of the heat equation.

- Coefficient of highest derivative (the mobility $h^3/3$) vanishes when $h = 0$, so the partial differential equation is degenerate at $h = 0$.

- Two important manifestations of this degeneracy are the existence of compactly supported mass preserving similarity solutions and "waiting time" behaviour