Problem set 10: Energy and uniquness

In 2019-20' iteration, Poisson's equation proofs not required (but not hard to do!)

PART A

In this set, you will show that solutions of ~~Poisson's equation,~~ the heat equation, and the wave equation are unique given certain boundary conditions. Consider the following systems defined on a volume $V \subseteq \mathbb{R}^3$ with boundary $S = \partial V$.

	~~Poisson's equation~~	Heat equation	Wave equation
Equation	$\nabla^2 u = f(\boldsymbol{x})$	$u_t = \kappa \nabla^2 u$	$u_{tt} = c^2 \nabla^2 u$
Initial condition	N/A	$u(\boldsymbol{x}, 0) = u_0$	$u(\boldsymbol{x}, 0) = U_0(\boldsymbol{x})$ $u_t(\boldsymbol{x}, 0) = V_0(\boldsymbol{x})$

The domains of the above functions are defined in the usual way, in consideration of $\boldsymbol{x} \in V$ and $t \geq 0$. Three possible boundary conditions, defined on $S = \partial V$, are now given for the above three systems.

Firstly, we may consider **Dirichlet** conditions:

$$u = F(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial V. \tag{D}$$

Or we may consider **Neumann** conditions:

$$\frac{\partial u}{\partial n} = G(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial V. \tag{N}$$

Or we may consider **mixed** conditions:

$$Au + B\frac{\partial u}{\partial n} = H(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \partial V, \tag{M}$$

where in the above, you may assume that A and B are constant, and A has the same sign as B.

1. For each of the ~~three~~ two equations above:
   (a) Begin by setting $w = u - v$ for two arbitrary solutions u and v. State the system of equations that w must satisfy.
   (b) Derive an expression for ~~the energy E (of Poisson's equation) or~~ the evolution of energy $E(t)$ (of the heat/wave equations).
   (c) Prove that solutions are either unique or defined up to a constant for the three boundary conditions of (D), (N), or (M).

   You may use the vector identity

$$\nabla \cdot (w_1 \nabla w_2) = \nabla w_1 \cdot \nabla w_2 + w_1 \nabla^2 w_2.$$